Finite expiry Russian options

J. J. Duistermaat [a], A. E. Kyprianou [a,*] and K. van Schaik [b]

[a] *Utrecht University, P.O.Box 80.010, 3500 TA, Utrecht, The Netherlands.*

[b] *J.W. Goethe-Universität Fb. Mathematik (Fach 187) 60054 Frankfurt am Main, Germany*

Abstract

We consider the Russian option introduced by Shepp and Shiryayev (1994, 1995) but with finite expiry and show that its space-time value function characterizes the unique solution to a free boundary problem. Further, using a method of randomization (or Canadization) due to Carr (1998) we produce a numerical algorithm for solving the aforementioned free boundary problem.

Key words: American options, Russian options, optimal stopping problem, Stefan boundary problem, Local time-space.
MSC(1991) 91B28, 35R35, 45G10

---

1 Introduction: the Russian option

Consider the Black-Scholes market. That is, a market with a risky asset S and a riskless bond, B. The bond evolves according to the dynamic

$$dB_t = rB_t dt$$

where $t \geq 0$ and $r > 0$. The risky asset is written as the process $S = \{S_t : t \geq 0\}$ where

$$S_t = s \exp\{\sigma W_t + \mu t\}$$

* Corresponding author.
Email addresses: duis@math.uu.nl (J. J. Duistermaat), kypriano@math.uu.nl (A. E. Kyprianou), schaik@math.uni-frankfurt.de (K. van Schaik).

where $s > 0$ is the initial value of S and $W = \{W_t : t \geq 0\}$ is a Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, P)$ satisfying the usual conditions. Suppose now that $\mathbb{P}_s$ is the risk-neutral measure for S under the assumption that $S_0 = s$. Recall that standard Black-Scholes theory dictates that this measure exists and is uniquely defined via a Girsanov change of measure such that

$$\{e^{-rt}S_t : t \geq 0\} \tag{1}$$

is a martingale. We shall denote $\mathbb{E}_s$ expectation under $\mathbb{P}_s$.

The Russian option with expiry $T < \infty$ is an American-type option with payout of the form

$$e^{-\alpha t}\left(\max\left\{m, \sup_{u\in[0,t]} S_u\right\}\right)$$

for $\alpha \geq 0$, $m > 0$ and $t \in [0,T]$. Introduced by Shepp and Shiryayev (1994, 1995) as being an option where one has 'reduced regret' because a minimum payout of m is guaranteed, this option can be considered to be something like an American-type lookback option.

Classical optimal stopping arguments for American-type options tell us that the value of this option is given by the process

$$V_t = \text{ess-sup}_{\tau\in\mathcal{T}_{t,T}}\mathbb{E}_s\left(e^{-r(\tau-t)}e^{-\alpha\tau}\max\left\{m, \sup_{u\in[0,t]} S_u\right\}\middle|\,\mathcal{F}_t\right) \tag{2}$$

where $\mathcal{T}_{t,T}$ is the set of $\mathbb{F}$-stopping times valued in $[t,T]$. Following the lead of Shepp and Shiryayev (1995), we use the fact that under $\mathbb{P}_s$, (1) is an exponential martingale and thus can be used to make a change of measure via

$$\left.\frac{\mathrm{d}\mathbf{P}_s}{\mathrm{d}\mathbb{P}_s}\right|_{\mathcal{F}_t} = \frac{e^{-rt}S_t}{s}.$$

Note that the process S solves

$$\frac{\mathrm{d}S_t}{S_t} = \sigma W_t^{\mathbf{P}} + \left(r + \sigma^2\right)\mathrm{d}t.$$

where $W^{\mathbf{P}} = \{W_t^{\mathbf{P}} : t \geq 0\}$, defined on $(\Omega, \mathcal{F}, \mathbb{F})$, is a standard Brownian motion under $\mathbf{P}$. Suppose that $-T_0$ is some arbitrary moment in the past before the contract was initiated ($T_0 > 0$). Define $\overline{S}_t = \sup_{u\in[-T_0,t]} S_u$ and assume that $\overline{S}_0$ is $\mathcal{F}_0$ measurable. With a slight abuse of notation, we can adapt the definition of the measure $\mathbf{P}_s(\cdot)$ to $\mathbf{P}_{m/s}(\cdot) = \mathbf{P}\left(\cdot|\overline{S}_0 = m, S_0 = s\right)$. In that case the value process of the option can be written more conveniently as

$$\left\{e^{-\alpha t} S_t v\left(\Psi_t, T-t\right) : t \in [0,T]\right\} \tag{3}$$

where $\Psi = \{\Psi_t = \overline{S}_t / S_t : t \in [0,T]\}$, $\mathbf{P}_{m/s}(\Psi_0 = m/s) = 1$ and for each $\psi = m/s \geq 1$,

$$v(\psi, u) = \sup_{\tau \in \mathcal{T}_{0,u}} \mathbf{E}_\psi\left(e^{-\alpha\tau}\Psi_\tau\right). \tag{4}$$

By writing stopping times in (2) in the form $\tau = t + \tau'$ for some $\tau' \in \mathcal{T}_{0,T-t}$ we see where the factor $e^{-\alpha t}$ in (3) comes from. It is also important to note in this case the well established fact from optimal stopping theory that stopping times in $\mathcal{T}_{t,T}$ which cannot be written in this way may be ignored as they do not offer an optimal strategy.

Effectively the change of measure has reduced an optimal stopping problem for two stochastic processes to that of one stochastic process, namely Ψ. For future reference, we shall also note that v may be represented in the following forms

$$v(\psi, u) = \sup_{\tau \in \mathcal{T}_{0,u}} \mathbf{E}\left(e^{-\alpha\tau}\frac{\overline{S}_\tau \vee \psi}{S_\tau}\middle| S_0 = 1, \overline{S}_0 = 1\right) \tag{5}$$

$$= \sup_{\tau \in \mathcal{T}_{0,u}} \mathbb{E}_1\left(e^{-(r+\alpha)\tau}(\overline{S}_\tau \vee \psi)\right). \tag{6}$$

Remark 1 At this point it is worth mentioning that, to some extent, one may consider the parameter α as superficial for the finite expiry option. Its original purpose for the perpetual case was essentially to justify the existence of a solution to the optimal stopping problem associated with the option; cf. Shepp and Shiryayev (1994, 1995). For finite expiry, the existence of a solution to the optimal stopping problem (4) is guaranteed even when $\alpha = 0$.

Clearly $v(\psi, u) \geq \psi$ for all $u \geq 0$. Standard theory of optimal stopping tells us that the optimal stopping time in (2) is given by

$$\eta_t^* = \inf\{s \geq t : v(\Psi_s, T-s) \leq \Psi_s\} \wedge (T-t).$$

[Here and throughout we work with the definition $\inf \emptyset = \infty$].

Again from the classical theory of optimal stopping (cf. Shiryayev (1978)) we know that for $t \in [0,T]$

$$\left\{e^{-\alpha s} v\left(\Psi_s, T-s\right) : s \in [t,T]\right\} \tag{7}$$

is a $\mathbf{P}$-supermartingale and that

$$\left\{e^{-\alpha(s\wedge\eta_t^*)} v\left(\Psi_{s\wedge\eta_t^*}, T-(s\wedge\eta_t^*)\right) : s \in [t,T]\right\} \tag{8}$$

is a $\mathbf{P}$-martingale.

Let us finish this section by making note of some analytical facts concerning the functions v and b.

Lemma 2 *We have the following properties of v and b.*

(i) The function v is convex in ψ.
(ii) The optimal stopping time in (5) may be identified in the form

$$\tau_\psi^* = \inf\{t \geq 0 : (\overline{S}_t \vee \psi)/S_t \geq b(u-t)\} \wedge u. \tag{9}$$

where $b : (0,\infty] \to [1,\infty)$ is defined by

$$b(u) = \inf\{\psi \geq 1 : v(\psi, u) = \psi\}$$

for $u > 0$.
(iii) The function v is jointly continuous in u and ψ and monotone non-decreasing in u.
(iv) The boundary b is monotone non-decreasing and continuous from the left. Further

$$\mathcal{C}(b) := \{(\psi, u) : 1 < \psi < b(u), u \in (0,T)\}$$

is open.
(v) The function v is strictly increasing in ψ.

Proof. (i) Since the object in the expectation in (6) is convex for each $\tau \in \mathcal{T}_{0,u}$ and convexity is preserved by integration and taking the supremum, convexity of v in ψ follows. Since $v \geq \psi$ the form of the stopping time follows.

(ii) As a partial step, let us prove that for each $u > 0$ the function $v(\psi, u) - \psi$ is non-increasing in ψ. To this end, consider $1 \leq \psi_1 \leq \psi_2 < \infty$. Write

$$\tau_\psi^* := \inf\{t \geq 0 : v((\overline{S}_t \vee \psi)/S_t, u-t) \leq (\overline{S}_t \vee \psi)/S_t\} \wedge u$$

for the optimal stopping time associated with the right hand side of (5) and note by optimality that

$$v(\psi_1, u) \geq \mathbb{E}_1(e^{-(r+\alpha)\tau^*_{\psi_2}}(\overline{S}_{\tau^*_{\psi_2}} \vee \psi_1)).$$

It now follows that

$$\begin{aligned} v(\psi_2, u) - v(\psi_1, u) &\leq \mathbb{E}_1(e^{-(r+\alpha)\tau^*_{\psi_2}}[(\overline{S}_{\tau^*_{\psi_2}} \vee \psi_2) - (\overline{S}_{\tau^*_{\psi_2}} \vee \psi_1)]) \\ &\leq \mathbb{E}_1(e^{-(r+\alpha)\tau^*_{\psi_2}}(\psi_2 - \psi_1)) \\ &\leq \psi_2 - \psi_1 \end{aligned}$$

from which the claimed monotonicity follows. We may now deduce from this monotonicity together with the fact that $v(\psi, u) \geq \psi$ that once the function $v(\psi, u)$ touches the diagonal ψ then it remains equal to ψ. Together with convexity this latter conclusion implies the statement in part (ii) of the Lemma.

(iii) From the definition (4), the function v is clearly monotone non-decreasing in u. The proof of joint continuity is a straightforward argument using dominated convergence. We leave the details to the reader and otherwise we refer to Van Schaik (2003).

(iv) The fact that b is non-decreasing follows from the monotonicity in u of v. The continuity of v implies that the function b is half-continuous from below (that is to say $\liminf_{v\to u} b(v) \geq b(u)$). Further, a monotone non-decreasing function is half-continuous from below if and only if it is left continuous. Next note that the region $\mathcal{C}(b)$ is open if and only if the function b is half-continuous from below, which is the case.

(v) Note that for $1 \leq x < y$, on the event $\{\overline{S}_t < y\}$ we have

$$\frac{\overline{S}_t \vee x}{S_t} < \frac{y}{S_t} = \frac{\overline{S}_t \vee y}{S_t}$$

and on the event $\{\overline{S}_t \geq y\}$ we have

$$\frac{\overline{S}_t \vee x}{S_t} = \frac{\overline{S}_t}{S_t} = \frac{\overline{S}_t \vee y}{S_t}.$$

It follows from the continuity of the paths of S that the stopping time τ^*_ψ is stochastically monotone in ψ and further, for any $\psi_1 > \psi_2 \geq 1$, $\mathbf{P}(\tau^*_{\psi_1} < \tau^*_{\psi_2}|S_0 = 1, \overline{S}_0 = 1) > 0$. Noting from (5) that

$$v(\psi, u) = \mathbf{E}(e^{-\alpha\tau^*_\psi}(\overline{S}_{\tau^*_\psi} \vee \psi)/S_{\tau^*_\psi}|\overline{S}_0 = 1, S_0 = 1),$$

the claim follows by taking account of the aforementioned stochastic monotonicity, part (ii) and the fact that $b(u-\cdot)$ is non-increasing. ■

2 Main results

In this paper we have two clear and simple goals. The first is to show that v may be characterized as the unique solution to a free boundary problem where the boundary turns out to be monotone and continuous and the second is to give a numerical algorithm for solving this free boundary problem. We summarize our results as follows.

Theorem 3 *The pair*

$$v:[1,\infty)\times[0,T]\to[1,\infty) \text{ and } b:[0,T]\to[1,\infty)$$

form the unique solution to the free boundary problem

$$\begin{aligned}
\left[\frac{\sigma^2}{2}\psi^2\frac{\partial^2 f}{\partial\psi^2}-r\psi\frac{\partial f}{\partial\psi}-\alpha f-\frac{\partial f}{\partial u}\right](\psi,u)&=0 \text{ on } \mathcal{C}(\varphi)\\
f(\psi,u)&=\psi \text{ on } \{[1,\infty)\times(0,T)\}\setminus\mathcal{C}(\varphi)\\
f(\psi,u)&>\psi \text{ on } \mathcal{C}(\varphi)\\
f(\psi,0)&=\psi \text{ for } \psi\in[1,\infty)\\
\frac{\partial f}{\partial\psi}\left(\varphi(u)^-,u\right)=\frac{\partial f}{\partial\psi}\left(\varphi(u)^+,u\right)&=1 \text{ for } u\in(0,T) \text{ (smooth pasting)}\\
\frac{\partial f}{\partial\psi}(1,u)&=0 \text{ for } u\in(0,T] \text{ (reflection)}
\end{aligned}\tag{10}$$

where $\mathcal{C}(\varphi)=\{(\psi,u):1<\psi<\varphi(u),\ u\in(0,T)\}$ for some monotone non-decreasing, bounded continuous function $\varphi:(0,T)\to(1,\infty)$.

[Note, the solution to the above free boundary problem is the pair (f,φ)]. The proof of this theorem will be the result of the combined conclusions found in the next section.

Next we turn to a numerical algorithm which serves as a good approximation to the solution to the above free boundary problem.

Algorithm 4 *For each $n\geq 1$, the solution to the free boundary problem may be approximated by*

$$v(\psi,u)=v^{(k,n)}(\psi) \text{ and } b(u)=\widetilde{\psi}_{(k,n)} \text{ if } u\in\left((k-1)\frac{T}{n},k\frac{T}{n}\right]$$

for $k = 1, ..., n$ *where the functions* $v^{(k,n)} : [1, \infty) \to [1, \infty)$ *and thresholds* $\widetilde{\psi}_{(k,n)}$ *relate to one another as follows. Let*

$$\lambda_n = \frac{n}{T} \text{ and } \delta_n = \frac{\lambda_n}{r + \alpha + \lambda_n}$$

and let $\beta_1 < \beta_2$ *be the two solutions to the quadratic equation*

$$\frac{\sigma^2}{2}\beta^2 - \left(r + \frac{\sigma^2}{2}\right)\beta - (\alpha + \lambda_n) = 0.$$

Then $v^{(0,n)}(\psi) = \psi$ *and* $\widetilde{\psi}_{(0,n)} = 1$ *and for* $k = 1, ..., n$

$$v^{(k,n)}(\psi) = \psi$$

if $\psi \geq \widetilde{\psi}_{(k,n)}$ *and otherwise, when* $\psi \in [\widetilde{\psi}_{(i-1,n)}, \widetilde{\psi}_{(i,n)}]$ *and* $i = 1, ..., k$,

$$\begin{aligned} v^{(k,n)}(\psi) = \psi^{\beta_1} & \left(c(1,i,k) + \sum_{m=1}^{k-i} a(m,i,k) \log(\psi)^m\right) \\ & + \psi^{\beta_2}\left(c(2,i,k) + \sum_{m=1}^{k-i} b(m,i,k) \log(\psi)^m\right) + \delta_n^{k-i+1}\psi \end{aligned}$$

All the constants in the latter are defined recursively and on account of the complexity of the recursion are given in the Appendix.

The algorithm will be dealt with in Section 4.

The formulation in Algorithm 4, although somewhat complicated, allows for one to construct quite precise numerical approximations to the free boundary problem in a package such as Mathematica for example. Indeed in the final section of this paper we give an exposition of the value functions as surfaces and optimal exercise boundaries produced by this algorithm with some indication of the efficiency of the programme.

Let us conclude this section by making some final remarks on our main results. To some extent our conclusions are not surprising on account of analogous results being available and well studied in the literature for American put options as well as the known results for perpetual Russian options. The reader is referred to Lamberton (1998), Karatzas and Shreve (1991), Myneni (1991), Shepp and Shiryayev (1994, 1995), Graversen and Peskir (1998), Kyprianou and Pistorius (2003), Avram *et al.* (2004a,b) and Carr (1998). None the less, until very recently, there was no literature concerning finite expiry Russian options. In parallel to the writing of this paper however, the authors learnt of the work of Peskir (2004b). This paper, which also handles the case of finite expiry Russian options, has some overlaps with the work presented here, but none the less deals with slightly different issues to the ones we address here. In particular, the main objective of Peskir's paper is to show how the function v may be expressed in terms of the optimal stopping boundary b which itself is the unique solution to a non-linear integral equation. Peskir (2004a) also deals with a similar representation for the American option.

3 Free boundary problem: proof of Theorem 3

We break the proof into a series of Lemmas which themselves are shared between two subsections dealing with existence and uniqueness respectively.

3.1 At most one solution to the free boundary problem

Lemma 5 *If a solution to (10) exists then it is equal to the pair (v, b). That is to say, (10) has at most one solution.*

Proof. Let (f, φ) be any solution to (10). Note that the corresponding region $\mathcal{C}(\varphi)$ is open for reasons given in the proof of Lemma 2. Define for each $t \in [0, T]$

$$\tau_t^{\mathcal{C}(\varphi)} = \inf\{s \geq t : \Psi_t \geq \varphi(T - t - s)\} \wedge (T - t)$$

and

$$\mathcal{L} = \frac{\sigma^2}{2}\psi^2 \frac{\partial^2}{\partial \psi^2} - r\psi \frac{\partial}{\partial \psi} - \alpha - \frac{\partial}{\partial u}.$$

Since in $\mathcal{C}(\varphi)$, f is $C^{2,1}$ (that is twice differentiable with continuous derivatives in the first parameter and once differentiable with continuous derivative in the second parameter) and $\mathcal{L}f = 0$, Itô's formula together with boundedness of φ easily yields that for each $t \in [0, T]$,

$$\left\{ e^{-\alpha(s \wedge \tau_t^{\mathcal{C}(\varphi)})} f\left(\Psi_{s \wedge \tau_t^{\mathcal{C}(\varphi)}}, T - (s \wedge \tau_t^{\mathcal{C}(\varphi)})\right) : s \in [t, T] \right\}$$

is a uniformly integrable $\mathbf{P}_\psi$-martingale for $\psi < \varphi(T - t)$.

Making use of a new generalized version of Itô's formula for continuous semi-martingales given in Theorem 3.1 of Peskir (2004c) (see also Eisenbaum (2000) and Fölmer et al. (1995) for developments prior to Peskir's formula and Peskir (2004d) and Eisenbaum (2004) for developments proceeding Peskir's formula) together with the fact that $\mathrm{d}\Psi_t = -\Psi_t(\sigma W_t^{\mathbf{P}} + rdt) + \mathrm{d}\overline{S}_t/S_t$ (cf. Shepp and Shiryayev (1995)) we may write

$$\begin{aligned}
&e^{\alpha t}\mathrm{d}[e^{-\alpha t} f(\Psi_t, T-t)] \\
&= \mathcal{L}f(\Psi_t, T-t)\mathrm{d}t - \sigma\Psi_t \frac{\partial f}{\partial\psi}(\Psi_t, T-t)\,\mathrm{d}W_t^{\mathbf{P}} \\
&\quad + \frac{1}{S_t}\frac{\partial f}{\partial\psi}(\Psi_t, T-t)\,\mathrm{d}\overline{S}_t \\
&\quad + \frac{1}{2}\left\{\frac{\partial f}{\partial\psi}\left(\Psi_t^+, T-t\right) - \frac{\partial f}{\partial\psi}\left(\Psi_t^-, T-t\right)\right\}\mathbf{1}_{(\Psi_t=\varphi(T-t))}\mathrm{d}L_t^{\varphi(T-t)} \qquad (11)
\end{aligned}$$

where $L_\cdot^{\varphi(T-\cdot)}$ is a version of the local time of Ψ at the curve $\varphi(T-\cdot)$. Note that Theorem 3.1 of Peskir (2004c) has three conditions which need checking. It can easily be confirmed that these conditions are automatically satisfied on account of the following facts: $\mathcal{L}f = 0$ on $\mathcal{C}(\varphi)$, $\mathcal{L}f = -(r+\alpha)\psi$ on $\mathrm{int}\{\mathcal{C}(\varphi)^c\}$, there is monotonicity of φ and the function f is assumed to have smooth pasting. We leave it as an exercise to confirm that the conditions of Peskir's theorem are fulfilled. The precise definition of $L_\cdot^{\varphi(T-\cdot)}$ is not of importance here since by the smooth pasting assumption we have that the coefficient of $\mathrm{d}L_t^{\varphi(T-t)}$ is zero. The reader is otherwise referred to Peskir (2004c,d) for further details of local time on a space-time curve (or local time-space).

We may now deduce from (11) that $e^{-\alpha t} f(\Psi_t, T-t)$ is the sum of a local martingale and a process of bounded variation which decreases (since $\mathcal{L}f = -(r+\alpha)f < 0$ on the complement of the continuation region). Since $\partial f/\partial\psi$ is bounded, the local martingale is in fact a martingale and we are left with the conclusion that, for each $t \in [0,T]$,

$$\left\{e^{-\alpha s} f(\Psi_s, T-s) : s \in [t,T]\right\} \qquad (12)$$

is a $\mathbf{P}$-supermartingale.

The martingale and supermartingale properties, (8) and (7) respectively, together with the facts that $f > \psi$ on $\mathcal{C}(\varphi)$ and that $f(\Psi_{\tau_t^{\mathcal{C}(\varphi)}}, T-\tau_t^{\mathcal{C}(\varphi)}) = \Psi_{\tau_t^{\mathcal{C}(\varphi)}}$ are now sufficient using classical methods to establish that $f = v$. Indeed, for each $t \in [0,T)$ using the supermartingale property,

$$\begin{aligned}
e^{-\alpha t} f(\psi, T-t) &\geq \sup_{\tau\in\mathcal{T}_{0,T-t}} \mathbf{E}_\psi\left(e^{-\alpha(t+\tau)} f(\Psi_\tau, T-(t+\tau))\right) \\
&\geq e^{-\alpha t} \sup_{\tau\in\mathcal{T}_{0,T-t}} \mathbf{E}_\psi\left(e^{-\alpha\tau}\Psi_\tau\right).
\end{aligned}$$

and further

$$
\begin{aligned}
e^{-\alpha t} f(\psi, T-t) &= \mathbf{E}\left(e^{-\alpha \tau_t^{\mathcal{C}(\varphi)}} f\left(\Psi_{\tau_t^{\mathcal{C}(\varphi)}}, T-\tau_t^{\mathcal{C}(\varphi)}\right) | \Psi_t = \psi\right) \\
&= \mathbf{E}\left(e^{-\alpha \tau_t^{\mathcal{C}(\varphi)}} \Psi_{\tau_t^{\mathcal{C}(\varphi)}} | \Psi_t = \psi\right) \\
&\leq e^{-\alpha t} \sup_{\tau \in \mathcal{T}_{0,T-t}} \mathbf{E}_\psi\left(e^{-\alpha \tau} \Psi_\tau\right)
\end{aligned}
$$

proving that $f(\psi, T-t) = \sup_{\tau \in \mathcal{T}_{0,T-t}} \mathbf{E}_\psi\left(e^{-\alpha \tau} \Psi_\tau\right)$. ■

3.2 At least one solution to the free boundary problem

It is clear now that we have one of the two directions in the proof of Theorem 3. The other direction requires more analysis which we now proceed with in the shape of further Lemmas. For clarity, recall that $\mathcal{C}(b) = \{(\psi, u) : 1 < \psi < b(u), u \in (0, T)\}$ which defines an open region; the so called continuation region.

Lemma 6 *In $\mathcal{C}(b)$, the function v is $C^{2,1}$ and satisfies $\mathcal{L}v = 0$ with*

$$
\frac{\partial v}{\partial \psi}(1, \cdot) := \frac{\partial v}{\partial \psi}\left(1^+, \cdot\right) = 0.
$$

Proof. The proof is based on an analogous result for American put options treated in Karatzas and Shreve (1991); see Theorem 2.7.7. It requires one to construct a local parabolic Dirichlet problem in any small open rectangular space-time region embedded in the continuation domain with boundary value v and reflection condition on $\psi = 1$ should this be included in the domain. Then a classic result giving unique stochastic representation identifies v as the solution to the Dirichlet problem and hence gives the smoothness properties of v. The details are left as an exercise for the reader. ■

Lemma 7 *The boundary $b(u)$ is bounded for each $u > 0$.*

Proof. The proof is trivial for the case when $\alpha > 0$ as the finite horizon value function is upper bounded by the infinite horizon problem which has a finite boundary. For $\alpha = 0$ there is no solution to the infinite horizon problem and hence we proceed with our proof for this case.

Since $\overline{S}_T$ is integrable (this follows from standard distributional properties of Brownian motion), dominated convergence together with (6) gives us that

$$
0 \leq v(\psi, u) - \psi \leq \mathbb{E}_1[(\overline{S}_{\tau_\psi^*} - \psi) \vee 0] \to 0 \tag{13}
$$

as ψ tends to infinity, where τ_ψ^* was given in Lemma 2. Using Lemma 6 together with the properties of v given in Lemma 2 we have that on $\mathcal{C}(b)$

$$\frac{\partial^2 v}{\partial \psi^2}(\psi, u) \geq \frac{2r}{\sigma^2}\psi^{-1}\frac{\partial v}{\partial \psi}(\psi, u) \geq 0. \tag{14}$$

Integration of the last inequality in ψ yields

$$\frac{\partial v}{\partial \psi}(\psi, u) \geq \frac{2r}{\sigma^2}\int_1^\psi \xi^{-1}\frac{\partial v}{\partial \psi}(\xi, u)\mathrm{d}\xi$$

in $\mathcal{C}(b)$.

Suppose now that $b(u) = \infty$ for some $u \in (0, T]$. For this u, the last inequality is valid for all $\psi > 1$. Also for this u, we know from (13) and convexity that $\partial v(\psi, u)/\partial\psi$ tends to one as ψ tends to infinity. However, these last two observations are incompatible because together they also imply that $\partial v(\psi, u)/\partial\psi$ tends to infinity as ψ tends to infinity. The contradiction lies in the assumed unboundedness of $b(u)$ so the proof is complete. ■

Lemma 8 *The value function v satisfies the boundary conditions.*

(i) *$v(\psi, u) \geq \psi$ for $u \in (0, T]$ and $v(\psi, 0) = \psi$.*
(ii) *For all $u \in (0, T]$ we have that $v(1, u) > 1$. In particular, from Lemma 6 it follows that for all $u \in (0, T]$,*

$$\frac{\partial v}{\partial \psi}(1, u) = 0.$$

Proof. (i) The first two conditions have been discussed in the introduction.

(ii) Suppose there exists a $u' > 0$ such that $v(1, u') = 1$. By the monotonicity in u established in Lemma 2, it follows that $v(1, u) = 1$ for all $u \leq u'$. This means that $v(\psi, u) = \psi$ for any such u and the optimal stopping time in (6) is to stop immediately. According to the supermartingale property given in (7) it now follows together with the representation of the value of the Russian option given in (3) that $e^{-(r+\alpha)t}\overline{S}_t$ is a $\mathbb{P}(\cdot|S_0 = \overline{S}_0 = s)$-supermartingale for $0 \leq t \leq u'$. Note now that the latter process has no martingale component and therefore must be a process which is monotone decreasing from an initial value s. In particular, it follows that

$$\sup_{0\leq t\leq u'} e^{-(r+\alpha)t}S_t = \sup_{0\leq t\leq u'} s \times e^{-(r+\alpha)t}e^{\sigma W_t + (r-\sigma^2/2)t} \leq s$$

where W is a $\mathbb{P}$ Brownian motion. However this leads to a contradiction since by the Law of the Iterated Logarithm for Brownian motion as $t \downarrow 0$, it follows that, given any $c > 0$, there exists a decreasing sequence of times $t_n(\omega) \downarrow 0$ such that $W_{t_n} > \sqrt{t_n} > ct_n$ and hence the supremum above is strictly greater than s. The consequence of this contradiction is that $v(1, u') > 1$. ∎

Lemma 9 *The function v exhibits the smooth pasting condition*

$$\frac{\partial v}{\partial \psi}\left(b(u)^+, u\right) = \frac{\partial v}{\partial \psi}\left(b(u)^-, u\right) = 1 \text{ for } u \in (0, T].$$

Proof. Since $v(\psi, u) = \psi$ for $\psi \geq b(u)$ and $v(\psi, u) > \psi$ for $1 \leq \psi < b(u)$ and $v(\cdot, u)$ is convex for each $u \in (0, T]$, it is trivial that

$$1 = \frac{\partial v}{\partial \psi}\left(b(u)^+, u\right) \geq \frac{\partial v}{\partial \psi}\left(b(u)^-, u\right). \tag{15}$$

It remains to prove then that

$$\frac{\partial v}{\partial \psi}\left(b(u)^-, u\right) \geq 1 \tag{16}$$

for all $u \in (0, T]$.

To this end, note from (6) that optimality implies that for $u \in (0, T)$

$$v(b(u), u) \geq \mathbb{E}_1\left(e^{-(r+\alpha)\tau^*_{b(u)-\epsilon}}(\overline{S}_{\tau^*_{b(u)-\epsilon}} \vee b(u))\right)$$

where $\epsilon > 0$ is small and $\tau^*_{b(u)-\epsilon}$ is given in Lemma 2. It now follows that

$$\begin{aligned}
&\frac{1}{\epsilon}(v(b(u), u) - v(b(u) - \epsilon, u)) \\
&\geq \frac{1}{\epsilon}\mathbb{E}_1\left(e^{-(r+\alpha)\tau^*_{b(u)-\epsilon}}[(\overline{S}_{\tau^*_{b(u)-\epsilon}} \vee b(u)) - (\overline{S}_{\tau^*_{b(u)-\epsilon}} \vee (b(u) - \epsilon))]\right).
\end{aligned}$$

It is easy to check that $[(\overline{S}_{\tau^*_{b(u)-\epsilon}} \vee b(u)) - (\overline{S}_{\tau^*_{b(u)-\epsilon}} \vee (b(u) - \epsilon))]/\epsilon$ is valued in $[0, 1]$ and hence regularity of Brownian paths together with dominated convergence implies that (16) holds. ∎

Lemma 10 *The boundary function b is continuous on $(0, T]$ and $b(0^+) = 1$.*

Proof. Left continuity has already been dealt with in Lemma 2. For right continuity, fix some $u_0 \in [0, T)$ and work with the convention that $b(0) := 1$.

We shall prove that $\lim_{u \downarrow u_0} b(u) = b(u_0)$ and hence in the case that $u_0 = 0$ this means that $b(0+) = 1$.

Since $v(b(u_0), u_0) = b(u_0)$ and $v(b(u), u) = b(u)$ we have the following integral formula

$$v(b(u_0), u) - b(u_0) = \int_{b(u_0)}^{b(u)} \left(1 - \frac{\partial v}{\partial \psi}(\xi, u)\right) \mathrm{d}\xi \qquad (17)$$

for each $u \in (u_0, T)$. Note that $b(u) \geq b(u_0)$. For any $b(u_0) \leq \xi \leq b(u)$, we have

$$1 - \frac{\partial v(\xi, u)}{\partial \xi} = \int_{\xi}^{b(u)} \frac{\partial^2 v(\eta, u)}{\partial \eta^2} \mathrm{d}\eta,$$

because $\partial v(\xi, u)/\partial \xi \to 1$ as $\xi \uparrow b(u)$. We now use the second order differential equation $\mathcal{L}v = 0$, in combination with $v(\eta, u) \geq \eta > 0$ and $\partial v(\eta, u)/\partial \eta \geq 0$, in order to obtain the estimate, for each $\xi \leq \eta < b(u)$,

$$\frac{\partial^2 v(\eta, u)}{\partial \eta^2} \geq \frac{2r}{\sigma^2 \eta} \frac{\partial v(\eta, u)}{\partial \eta} \geq \frac{2r}{\sigma^2 \eta} \frac{\partial v(\xi, u)}{\partial \xi},$$

where we have used the convexity of $\psi \mapsto v(\psi, u)$ in the second inequality. This leads, with the notation

$$w(\xi, u) = \frac{2r}{\sigma^2} \log(b(u)/\xi),$$

to the estimates

$$1 - \frac{\partial v(\xi, u)}{\partial \xi} \geq w(\xi, u) \frac{\partial v(\xi, u)}{\partial \xi},$$

hence

$$\frac{\partial v(\xi, u)}{\partial \xi} \leq 1/(1 + w(\xi, u)),$$

and therefore

$$1 - \frac{\partial v(\xi, u)}{\partial \xi} \geq w(\xi, u)/(1 + w(\xi, u)),$$

which in turn implies that

$$v(b(u_0), u) - v(b(u_0), u_0) \geq \int_{b(u_0)}^{b(u)} \frac{w(\xi, u)}{1 + w(\xi, u)} \mathrm{d}\xi.$$

Suppose now that $b(u_0^+) > b(u_0)$. Because the left hand side converges to zero, this would imply that

$$0 = \int_{b(u_0)}^{b(u_0^+)} \frac{w(\xi, u_0)}{1 + w(\xi, u_0)} \mathrm{d}\xi,$$

in which case

$$w(\xi, u_0) = \frac{2r}{\sigma^2} \log(b(u_0^+)/\xi) > 0 \quad \text{when} \quad 0 < \xi < b(u_0^+).$$

This contradiction proves the right continuity and hence continuity of the function b. ■

4 Canadization

Carr (1998) proposes a novel and yet simple method of approximating the price of the finite expiry American put at time T via a method of randomization or Canadization as he calls it. The idea is quite simple. As a first approximation, one may consider randomizing the expiry date, T, of the option by an independent exponential distribution having mean T and forcing the American put claim should the option expire at the end of this exponential time. The logic behind this randomization is that the free boundary problem is converted from a time variant one to a time invariant one as a consequence of the lack of memory property; if the holder has not yet exercised, then there is still an exponential time remaining. It is reasonably intuitive to see that that the effect of this randomization is to convert the parabolic free boundary problem associated with the American put to an elliptic free boundary problem. The latter being explicitly solvable.

A natural generalization of this idea which Carr further pursues is to replace the exponential distribution by a sum of n independent exponential distributions, each having mean T/n so that the expectation of the sum is T and again forcing the American put claim should the option expire at this random time. We shall refer to this as an *n-th order randomization*. Suppose we denote each of these exponentials by $\mathbf{e}_{i,n}$ then by the Law of Large Numbers it follows that

$$\sum_{i=1}^{n} \mathbf{e}_{i,n} = \sum_{i=1}^{n} \frac{T}{n} \left[\frac{n}{T} \mathbf{e}_{i,n} \right] \to T$$

almost surely. This shows that if one can solve the optimal stopping problem with a randomized expiry according to the independent distribution $\sum_{i=1}^{n} \mathbf{e}_{i,n}$ then to some extent for large n one has a good approximation to the finite expiry case; and hence by the previous section a good approximation to the associated free boundary value problem. Carr (1998) makes good of this approximation and provides an explicit expression for the case of the n-th order randomization of the American put option. This expression is the consequence of a sequence of iterated elliptic free boundary problems.

In this section we formulate the problem of the n-th order randomization for the Russian option and show that like Carr's results for the American put the resulting approximation is represented by the solution to an iterated system of elliptic free boundary problems which we solve explicitly. This solution leads to Alogrithm 4.

4.1 n-th order randomization

The n-th order randomization which approximates the function $v(\psi, T)$ is the solution to the optimal stopping problem

$$v^{(n,n)}(\psi) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathcal{E} \otimes \mathbf{E}_\psi \left(e^{-\alpha(\tau \wedge \Theta_{n,n})} \Psi_{\tau \wedge \Theta_{n,n}} \right)$$

where under the measure $\mathcal{P}$ (having expectation operator $\mathcal{E}$), $\Theta_{n,n}$ is the sum of n independent exponential random variables $\{\mathbf{e}_{i,n} : i = 1, ..., n\}$ with parameter

$$\lambda_n := n/T$$

and $\mathcal{T}_{0,\infty}$ is the set of $\mathbb{F}$-stopping times valued in $[0, \infty)$. The choice of notation $v^{(n,n)}(\psi)$ and $\Theta_{n,n}$ will become apparent in a moment.

Lemma 11 *The function $v^{(n,n)}(\psi)$ is the final step in the recursion*

$$v^{(0,n)}(\psi) = \psi \text{ and}$$
$$v^{(k,n)}(\psi) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbf{E}_\psi \left(e^{-(\alpha+\lambda_n)\tau} \Psi_\tau + \lambda_n \int_0^\tau e^{-(\alpha+\lambda_n)s} v^{(k-1,n)}(\Psi_s) ds \right)$$

for $k = 1, ..., n$.

Proof. Suppose that under measure $\mathcal{P}$ we now define

$$\Theta_{k,n} = \sum_{i=k}^{n} \mathbf{e}_{i,n},$$

We have

$$
\begin{aligned}
&v^{(n,n)}(\psi)\\
&= \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathcal{E}\otimes\mathbf{E}_\psi\left(e^{-\alpha(\tau\wedge\Theta_{n,n})}\Psi_{\tau\wedge\Theta_{n,n}}\left(\mathbf{1}_{(\tau\le\Theta_{1,n})}+\mathbf{1}_{(\tau>\Theta_{1,n})}\right)\right)\\
&= \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathcal{E}\otimes\mathbf{E}_\psi\left(e^{-\alpha\tau}\Psi_\tau\mathbf{1}_{(\tau\le\Theta_{1,n})}\right.\\
&\quad\left.+\mathbf{1}_{(\tau>\Theta_{1,n})}e^{-\alpha\Theta_{1,n}}e^{-\alpha((\tau-\Theta_{1,n})\wedge\Theta_{n-1,n})}\Psi_{\Theta_{1,n}+((\tau-\Theta_{1,n})\wedge\Theta_{n-1,n})}\right)\\
&= \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathcal{E}\otimes\mathbf{E}_\psi\left(e^{-\alpha\tau}\Psi_\tau\mathbf{1}_{(\tau\le\Theta_{1,n})}\right.\\
&\quad\left.+\mathbf{1}_{(\tau>\Theta_{1,n})}e^{-\alpha\Theta_{1,n}}\mathcal{E}\otimes\mathbf{E}_\psi^{\mathcal{F}_{\Theta_{1,n}}}\left[e^{-\alpha((\tau-\Theta_{1,n})\wedge\Theta_{n-1,n})}\Psi_{\Theta_{1,n}+((\tau-\Theta_{1,n})\wedge\Theta_{n-1,n})}\right]\right)
\end{aligned}
$$

where in the third equality $\mathcal{F}_{\Theta_{1,n}} = \sigma(\mathcal{F}_t : t\le\Theta_{1,n})$. The Strong Markov Property together with the dynamic programming principle and lack of memory property now gives us

$$
\begin{aligned}
&v^{(n,n)}(\psi)\\
&= \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathcal{E}\otimes\mathbf{E}_\psi\left(e^{-\alpha\tau}\Psi_\tau\mathbf{1}_{(\tau\le\Theta_{1,n})}\right.\\
&\quad\left.+\mathbf{1}_{(\tau>\Theta_{1,n})}e^{-\alpha\Theta_{1,n}}\sup_{\tau\in\mathcal{T}_{0,\infty}}\mathcal{E}\otimes\mathbf{E}_{\Psi_{\Theta_{1,n}}}\left[e^{-\alpha(\tau\wedge\Theta_{n-1,n})}\Psi_{\tau\wedge\Theta_{n-1,n}}\right]\right).
\end{aligned}
$$

Now writing

$$
v^{(n-1,n)}(\psi) = \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathcal{E}\otimes\mathbf{E}_\psi\left[e^{-\alpha(\tau\wedge\Theta_{n-1,n})}\Psi_{\tau\wedge\Theta_{n-1,n}}\right]
$$

it follows that

$$
\begin{aligned}
v^{(n,n)}(\psi) &= \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathcal{E}\otimes\mathbf{E}_\psi\left(e^{-\alpha\tau}\Psi_\tau\mathbf{1}_{(\tau\le\Theta_{1,n})}+\mathbf{1}_{(\tau>\Theta_{1,n})}e^{-\alpha\Theta_{1,n}}v^{(n-1,n)}(\Psi_{\Theta_{1,n}})\right)\\
&= \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathbf{E}_\psi\left(e^{-(\alpha+\lambda_n)\tau}\Psi_\tau+\lambda_n\int_0^\tau e^{-(\alpha+\lambda_n)s}v^{(n-1,n)}(\Psi_s)ds\right).
\end{aligned}
$$

Iterating this argument and noting that

$$
v^{(1,n)}(\psi) = \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathbf{E}_\psi\left(e^{-(\alpha+\lambda_n)\tau}\Psi_\tau+\lambda_n\int_0^\tau e^{-(\alpha+\lambda_n)s}\Psi_s ds\right)
$$

the proof is complete. ■

Remark 12 Using similar reasoning it is easy to deduce that we may also identify

$$v^{(k,n)}(\psi) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathcal{E} \otimes \mathbf{E}_\psi \left[e^{-\alpha(\tau \wedge \Theta_{k,n})} \Psi_{\tau \wedge \Theta_{k,n}} \right]$$

for each $1 \leq k < n$.

Remark 13 Roughly speaking, by considering the case $k = 1$, one may establish that $v^{(1,n)}(\psi)$ is a convex function associated to which is the value $\widetilde{\psi}_{(1,n)} > 1 =: \widetilde{\psi}_{(0,n)}$ such that the optimal stopping time in the definition of $v^{(1,n)}(\psi)$ is given by

$$\tau^{(1,n)} = \inf\{t \geq 0 : \Psi_t \geq \widetilde{\psi}_{(1,n)}\}.$$

Indeed, similar conclusions were drawn for the 1-st order randomization in Kyprianou and Pistorius (2003) for the case that $\alpha = 0$ and $n = 1$. Proceeding to the cases $n \geq k \geq 2$, using an iteration which takes advantage of the convexity of $v^{(k-1,n)}(\psi)$ it is possible to show that, the optimal stopping time in the definition of $v^{(k,n)}(\psi)$ takes the form

$$\tau^{(k,n)} = \inf\{t \geq 0 : \Psi_t \geq \widetilde{\psi}_{(k,n)}\}$$

for some $\widetilde{\psi}_{(k,n)} > \widetilde{\psi}_{(k-1,n)}$.

4.2 Discrete Stefan system

The goal of this subsection is to show that the discrete Stefan system, defined below, has exactly one solution which can be described explicitly. Further, we will show that any solution must correspond to the n-th order randomization $\{v^{(k,n)} : k = 0, ..., n\}$ and hence the justification of Algorithm 4 will follow.

Definition 14 (Discrete Stefan system) *We say the pair* $\{f^{(k,n)} : k = 0, ..., n\}$ *and* $\{\varphi^{(k,n)} : k = 0, ..., n\}$ *where*

$$f^{(k,n)} : [1, \infty) \to [1, \infty) \text{ and } \varphi^{(k,n)} \geq 1$$

solves the discrete Stefan system if $f^{(0,n)}(\psi) = \psi$ *and* $\varphi^{(0,n)} = 1$ *and for* $k = 1, ..., n$ *we have*

$$\frac{\sigma^2}{2}\psi^2 \frac{\mathrm{d}^2 f^{(k,n)}}{\mathrm{d}\psi^2}(\psi) - r\psi \frac{\mathrm{d}f^{(k,n)}}{\mathrm{d}\psi}(\psi) - (\alpha + \lambda_n) f^{(k,n)}(\psi) = -\lambda_n f^{(k-1,n)}(\psi)$$

$$\text{for } 1 < \psi < \varphi^{(k,n)} \text{ and}$$
$$f^{(k,n)}(\psi) = \psi \text{ for } \psi \geq \varphi^{(k,n)}. \tag{18}$$

Furthermore, for $k = 1, ..., n$,

$$\lim_{\psi \downarrow 1} \frac{\mathrm{d} f^{(k,n)}}{\mathrm{d}\psi}(\psi) = 0, \quad \lim_{\psi \uparrow \varphi^{(k,n)}} \frac{\mathrm{d} f^{(k,n)}}{\mathrm{d}\psi}(\psi) = 1 \text{ and } \lim_{\psi \uparrow \varphi^{(k,n)}} f^{(k,n)}(\psi) = \varphi^{(k,n)}.$$

The following theorem is proved at the end of this subsection.

Theorem 15 *A unique solution exists to the discrete Stefan system. In addition, this unique solution satisfies* $\varphi^{(n,n)} > ... > \varphi^{(0,n)} = 1$ *and for all* $k = 1, ..., n$

$$\frac{\mathrm{d} f^{(k,n)}}{\mathrm{d}\psi}(\psi) < 1 \text{ and } f^{(k,n)}(\psi) > \psi$$

when $1 \leq \psi < \varphi^{(k,n)}$. *Further the unique solution may be identified by* $f^{(k,n)} = v^{(k,n)}$ *for all* $k = 0, ..., n$ *and hence the thresholds* $\{\widetilde{\psi}_{(k,n)} : k = 1, ..., n\}$ *referred to in Remark 13 are precisely* $\{\varphi^{(k,n)} : k = 0, ..., n\}$.

Remark 16 Equation (18) can be rewritten

$$\frac{\sigma^2}{2}\psi^2 f^{(k,n)\prime\prime}(\psi) - r\psi f^{(k,n)\prime}(\psi) - \alpha f^{(k,n)}(\psi) = \frac{f^{(k,n)}(\psi) - f^{(k-1,n)}(\psi)}{\frac{T}{n}}. \tag{19}$$

For partial differential equations, such as the Stefan problem with solution v from the previous section, one has the so-called method of lines as a method of approximation. In this case, it could consist of putting a uniform grid on some fixed interval $[0, T]$ with distance T/n and approximate the derivative in the T-direction by its difference quotient

$$\frac{v(\psi, k\frac{T}{n}) - v(\psi, (k-1)\frac{T}{n})}{\frac{T}{n}}$$

such that the pde is broken up in a set of differential equations. Note that if we associate $f^{(k,n)}(\psi)$ with $v(\psi, k\frac{T}{n})$ this method precisely results in the set of differential equations of the form (19).

One important difference between the discrete Stefan system we deduced and this method of lines, is that it's not a priori clear how to deal with the fact that the boundary of the definition area of the pde is a curve rather than fixed.

Next we identify the promised explicit solution to the discrete Stefan system.

Lemma 17 *The pair* $\{v^{(k,n)} : k = 0, ..., n\}$ *and* $\{\widetilde{\psi}_{(k,n)} : k = 0, ..., n\}$ *are given by* $\widetilde{\psi}_{(0,n)} = 1$,

$$v^{(k,n)}(\psi) = \begin{cases} \psi \text{ for } \psi \geq \widetilde{\psi}_{(k,n)} \\ \\ \psi^{\beta_1}\left(c(1,i,k) + \sum_{m=1}^{k-i} a(m,i,k)\log(\psi)^m\right) \\ +\psi^{\beta_2}\left(c(2,i,k) + \sum_{m=1}^{k-i} b(m,i,k)\log(\psi)^m\right) + \delta^{k-i+1}\psi \\ \text{for } \psi \in [\widetilde{\psi}_{i-1}, \widetilde{\psi}_{(i,n)}] \text{ and } i = 1, ..., k \end{cases} \tag{20}$$

where $\beta_1 < \beta_2$ *are the two solutions to the quadratic equation*

$$\frac{\sigma^2}{2}\beta^2 - \left(r + \frac{\sigma^2}{2}\right)\beta - (\alpha + \lambda_n) = 0,$$

$\delta = \lambda_n/(r+\alpha+\lambda_n)$ *and, on account of their complexity, the constants* $a(.,.,.)$, $b(.,.,.)$, $c(.,.,.)$ *and the thresholds* $\widetilde{\psi}_{(k,n)}$ *are given in the Appendix. This pair solves, for* $k = 1, ..., n$,

$$\frac{\sigma^2}{2}\psi^2\frac{\mathrm{d}^2 v^{(k,n)}}{\mathrm{d}\psi^2}(\psi) - r\psi\frac{\mathrm{d}v^{(k,n)}}{\mathrm{d}\psi}(\psi) - (\alpha+\lambda_n)v^{(k,n)}(\psi) = -\lambda_n v^{(k-1)}(\psi)$$
$$\text{for } 1 < \psi < \widetilde{\psi}_{(k,n)}$$
$$v^{(k,n)}(\psi) = \psi \text{ for } \psi \geq \widetilde{\psi}_{(k,n)},$$

and furthermore

$$\lim_{\psi\downarrow 1}\frac{\mathrm{d}v^{(k,n)}}{\mathrm{d}\psi}(\psi) = 0, \quad \lim_{\psi\uparrow\widetilde{\psi}_{(k,n)}}\frac{\mathrm{d}v^{(k,n)}}{\mathrm{d}\psi}(\psi) = 1 \text{ and } \lim_{\psi\uparrow\widetilde{\psi}_{(k,n)}} v^{(k,n)}(\psi) = \widetilde{\psi}_{(k,n)}.$$

Proof. Given that we have identified the pair $\{v^{(k,n)} : k = 0, ..., n\}$ and $\{\widetilde{\psi}_{(k,n)} : k = 0, ..., n\}$ as the unique solution, it sufficies to check that the right hand side of (20) solves the discrete Stefan system. Sadly there is no elegant proof of this and a manual computation is the quickest way of establishing this result. In the computation, one should use the result of Theorem 15 to ensure that the defining equation for $\widetilde{\psi}_{(k,n)}$ (equation (25) in the Appendix) indeed has a unique solution that is strictly bigger than $\widetilde{\psi}_{(k-1,n)}$. Otherwise there is nothing special involved in the calculation other than the need for endurance. We leave the proof to the reader. ■

Returning to the proof of Theorem 15, we first need the following result which, as we shall see, easily resolves the issue of existence together with some of the conditions stipulated in Theorem 15. These latter conditions turn out to be crucial in order to prove that the unique solution is precisely $\{v^{(k,n)} : k = 0, ..., n\}$.

Theorem 18 *Fix $\lambda > 0$. Suppose that the function $f : [1, \infty) \to [1, \infty)$ satisfies the following*

(i) *there exists a $b \geq 1$ such that $f(\psi) = \psi$ for all $\psi \geq b$ and*
(ii) *if $b > 1$ then $f'(\psi) < 1$ for all $1 \leq \psi < b$.*

Then the system

$$\frac{\sigma^2}{2}\psi^2 u''(\psi) - r\psi u'(\psi) - (\alpha + \lambda)u(\psi) = -\lambda f(\psi) \qquad \text{for } 1 < \psi < c \quad (21)$$

and $u(\psi) = \psi$ for $\psi \geq c$, with the boundary conditions

$$\lim_{\psi \downarrow 1} u'(\psi) = 0, \qquad \lim_{\psi \uparrow c} u(\psi) = c \text{ and } \lim_{\psi \uparrow c} u'(\psi) = 1,$$

has at least one pair (u, c) with $c > 1$ as its solution. Every possible solution (u, c) possesses the property $u'(\psi) < 1$ for all $1 \leq \psi < c$ and either we have $c \geq b$ or for all $c \leq \psi \leq b$

$$f(\psi) \leq \frac{r + \alpha + \lambda}{\lambda}\psi.$$

Proof. From general theory of differential equations it follows that every solution of (21) can be written in the form

$$u(\psi) = a\psi^{\beta_2} + d\psi^{\beta_1} + u_0(\psi) \quad (22)$$

with

$$u_0(\psi) = \frac{-2\lambda}{\sigma^2(\beta_2 - \beta_1)} \int_1^{\psi} \left(\left(\frac{\psi}{\xi}\right)^{\beta_2} - \left(\frac{\psi}{\xi}\right)^{\beta_1} \right) \frac{f(\xi)}{\xi} \, \mathrm{d}\xi,$$

$\beta_1 < \beta_2$ solutions to the quadratic equation

$$\frac{\sigma^2}{2}\beta(\beta - 1) - r\beta - (\alpha + \lambda) = 0$$

and the a and d free constants.

Now pick any $x > 1$. We can choose the constants $a = a_x$ and $d = d_x$ now such that two out of three boundary conditions are satisfied: $u(x) = x$ and $u'(x) = 1$. A straightforward calculation shows that the appropriate choices are

$$a_x = \frac{2\lambda}{\sigma^2(\beta_2 - \beta_1)} \int_1^x \xi^{-\beta_2} \frac{f(\xi)}{\xi} \, \mathrm{d}\xi + \frac{1-\beta_1}{\beta_2 - \beta_1} x^{1-\beta_2}$$

and

$$d_x = \frac{-2\lambda}{\sigma^2(\beta_2 - \beta_1)} \int_1^x \xi^{-\beta_1} \frac{f(\xi)}{\xi} \, \mathrm{d}\xi + \frac{\beta_2 - 1}{\beta_2 - \beta_1} x^{1-\beta_1}.$$

The solution u we're looking for thus must be a member of the family of functions $u_x(\psi) = a_x \psi^{\beta_2} + d_x \psi^{\beta_1} + u_0(\psi)$. We define the following operator:

$$F_f : x \mapsto u_x'(1)$$

and note that we're looking for a root of this operator, meaning a $c \geq 1$ such that $F_f(c) = 0$. Once again straightforward calculation shows

$$\begin{aligned} F_f(x) = & \frac{2\lambda}{\sigma^2(\beta_2 - \beta_1)} \int_1^x \left(\beta_2 \xi^{-\beta_2} - \beta_1 \xi^{-\beta_1}\right) \frac{f(\xi)}{\xi} \, \mathrm{d}\xi + \beta_2 \frac{1-\beta_1}{\beta_2 - \beta_1} x^{1-\beta_2} \\ & + \beta_1 \frac{\beta_2 - 1}{\beta_2 - \beta_1} x^{1-\beta_1}. \end{aligned} \tag{23}$$

Now, c is a root of F_f if and only if the pair (u_c, c) is a solution as meant in theorem 18. It is easy to check that $F_f(1) = 1$ and using $f(\xi) = \xi$ for all $\xi \geq b$

$$\lim_{x \to \infty} F_f(x) = \lim_{x \to \infty} C x^{1-\beta_1}$$

with

$$C = \frac{2\beta_1(r + \alpha)}{(1-\beta_1)(\beta_2 - \beta 1)\sigma^2} < 0 \text{ and } \beta_1 < 0,$$

so that we can be sure that there exists at least one root of F_f on $(1, \infty)$ and therefore a solution (u, c).

To prove that $u'(\psi) < 1$ for all $1 \leq \psi < c$, note that the representation of u, given by (22), indicates that in fact u is a C^∞-function on the interval $(1, c)$.

With this in mind we do the following. Define $\xi(\psi) = u'(\psi)$ and suppose that ξ attains a maximum in some ψ_0, where $1 \leq \psi_0 < c$. As a consequence we have that $\xi'(\psi_0) = 0$ and $\xi''(\psi_0) \leq 0$.

By differentiating the de in (21) once, we see that this boils down to

$$\frac{\sigma^2}{2}\psi_0^2\xi''(\psi_0) - (r + \alpha + \lambda)\xi(\psi_0) = -\lambda f'(\psi_0)$$

which leads to

$$(r + \alpha + \lambda)\xi(\psi_0) = \frac{\sigma^2}{2}\psi_0^2\xi''(\psi_0) + \lambda f'(\psi_0) \leq \lambda,$$

and

$$\xi(\psi_0) = \frac{\lambda}{r + \alpha + \lambda} < 1.$$

So the only possibility for u' to be bigger than or equal to 1 somewhere on the interval $[1, c)$, with $u'(c) = 1$ in mind and avoiding reaching a maximum, is when there exists a ψ_0 such that for all $\psi_0 < \psi < c$ we have $u'(\psi) = 1$. For such a ψ we would have $\xi'(\psi) = \xi''(\psi) = 0$ and the above reasoning would still be valid, again leading to $\xi(\psi) = u'(\psi) < 1$. Thus $u'(\psi) < 1$ for all $1 \leq \psi < c$ is proven.

For the last part, suppose that $c < b$. From $u'(\psi) < 1$ for all $1 \leq \psi < c$ and $u'(c) = 1$ it follows that $u''(c-) \geq 0$. Using this with $u'(c) = 1$ and $u(c) = c$ in taking the limit $\psi \uparrow c$ in (21) leads to

$$f(c) \leq \frac{r + \alpha + \lambda}{\lambda}c.$$

Combining this with $f' \leq 1$ we arrive at

$$f(\psi) \leq \frac{r + \alpha + \lambda}{\lambda}\psi$$

for all $c \leq \psi \leq b$. ∎

Proof of Theorem 15. Note that the matter of existence is covered by inductively applying the result from Theorem 18, starting from $f^{(0,n)}(\psi) = \psi$. Now the proof breaks up in two parts. First we use the properties that a solution to the discrete Stefan system has according to Theorem 18 to prove that every possible solution may be identified by $f^{(k,n)} = v^{(k,n)}$, obviously implying uniqueness together with the other properties mentioned in Theorem 15, except for the fact that the thresholds $\varphi^{(0,n)}, ..., \varphi^{(n,n)}$ are strictly increasing, which will be dealt with in the second part of this proof.

For the first part, note that it is clear that $f^{(0,n)}(\psi) = v^{(0,n)}(\psi)$ and properties (i) and (ii) in Theorem 18 are satisfied. Next suppose that we have established that $f^{(k-1,n)}(\psi) = v^{(k-1,n)}(\psi)$ and such that properties (i) and (ii) of Theorem 18 hold. Then Theorem 18 tells us that a solution $f^{(k,n)}(\psi), \varphi^{(k,n)}$ exists and has the stated properties. To finish this part, we must henceforth show that $f^{(k,n)}(\psi) = v^{(k,n)}(\psi)$.

To this end we make an application of Itô's formula to the process

$$\left\{e^{-\alpha t} f^{(k,n)}(\Psi_t) : t \geq 0\right\}.$$

Noting that $f^{(k,n)}$ is sufficiently smooth to use the standard version of Itô's formula, that is to say it is smooth everywhere except at $\varphi^{(k,n)}$ where it is C^1 (cf. Karatzas and Shreve p215 for example), we have

$$\begin{aligned}
&\mathrm{d}[e^{-(\alpha+\lambda_n)t} f^{(k,n)}(\Psi_t)] + \lambda_n e^{-(\alpha+\lambda_n)t} f^{(k-1,n)}(\Psi_t)\, dt \\
&= e^{-(\alpha+\lambda_n)t}\left[\frac{\sigma^2}{2}\psi^2 \frac{\mathrm{d}^2 f^{(k,n)}}{\mathrm{d}\psi^2} - r\psi \frac{\mathrm{d} f^{(k,n)}}{\mathrm{d}\psi} - (\alpha+\lambda_n) f^{(k,n)} + \lambda_n f^{(k-1,n)}\right](\Psi_t)\, dt \\
&\quad - e^{-(\alpha+\lambda_n)t}\sigma \Psi_t \frac{\mathrm{d} f^{(k,n)}}{\mathrm{d}\psi}(\Psi_t)\, \mathrm{d}W_t^{\mathbf{P}}.
\end{aligned}$$

Noting that the first derivative of $f^{(k,n)}$ is bounded, and that from the conclusions of Theorem 18 we have

$$\left[\frac{\sigma^2}{2}\psi^2 \frac{\mathrm{d}^2 f^{(k,n)}}{\mathrm{d}\psi^2} - r\psi \frac{\mathrm{d} f^{(k,n)}}{\mathrm{d}\psi} - (\alpha+\lambda_n) f^{(k,n)} + \lambda_n f^{(k-1,n)}\right](\psi) \leq 0$$

both if $\varphi^{(k,n)} \geq \varphi^{(k-1,n)}$ and $\varphi^{(k,n)} < \varphi^{(k-1,n)}$, we deduce that

$$\left\{e^{-(\alpha+\lambda_n)t} f^{(k,n)}(\Psi_t) + \lambda_n \int_0^t e^{-(\alpha+\lambda_n)s} f^{(k-1,n)}(\Psi_s)\, \mathrm{d}s : t \leq \tau^{(k,n)}\right\}$$

is a martingale where $\tau^{(k,n)} = \inf\{t \geq 0 : \Psi_t \geq \varphi^{(k,n)}\}$ and

$$\left\{e^{-(\alpha+\lambda_n)t} f^{(k,n)}(\Psi_t) + \lambda_n \int_0^t e^{-(\alpha+\lambda_n)s} f^{(k-1,n)}(\Psi_s)\, \mathrm{d}s : t \geq 0\right\}$$

is a supermartingale.

Once again we appeal to classical arguments from the theory of optimal stopping to finish the proof. That is, using the supermartingale property together with the lower bound on $f^{(k,n)}$ we have with the help of Doob's Optional Stopping Theorem

$$\begin{aligned} f^{(k,n)}(\psi) \geq & \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbf{E}_\psi \left(e^{-(\alpha+\lambda_n)\tau} f^{(k,n)}(\Psi_\tau) \right. \\ & \left. + \lambda_n \int_0^\tau e^{-(\alpha+\lambda_n)s} f^{(k-1,n)}(\Psi_s)\,\mathrm{d}s \right) \\ \geq & \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbf{E}_\psi \left(e^{-(\alpha+\lambda_n)\tau} \Psi_\tau + \lambda_n \int_0^\tau e^{-(\alpha+\lambda_n)s} f^{(k-1,n)}(\Psi_s)\, ds \right). \end{aligned}$$

On the other hand, by the martingale property, we also have that

$$\begin{aligned} f^{(k,n)}(\psi) = & \mathbf{E}_\psi \left(e^{-(\alpha+\lambda_n)\tau^{(k,n)}} f^{(k,n)}(\Psi_{\tau^{(k,n)}}) \right. \\ & \left. + \lambda_n \int_0^{\tau^{(k,n)}} e^{-(\alpha+\lambda_n)s} f^{(k-1,n)}(\Psi_s)\,\mathrm{d}s \right) \\ = & \mathbf{E}_\psi \left(e^{-(\alpha+\lambda_n)\tau^{(k,n)}} \Psi_{\tau^{(k,n)}} + \lambda_n \int_0^{\tau^{(k,n)}} e^{-(\alpha+\lambda_n)s} f^{(k-1,n)}(\Psi_s)\,\mathrm{d}s \right) \end{aligned}$$

showing that $f^{(k,n)}(\psi) = v^{(k,n)}(\psi)$ and that $\varphi^{(k,n)}$ is the optimal threshold.

Now for a proof that $\varphi^{(n,n)} > ... > \varphi^{(0,n)}$, again by induction. It is straightforward that $\varphi^{(1,n)} > \varphi^{(0,n)} = 1$. Suppose that $\varphi^{(k-1,n)} < \varphi^{(k,n)}$. From Theorem 18 we have for all $\varphi^{(k-1,n)} \leq \psi < \varphi^{(k,n)}$ that $f^{(k,n)}(\psi) > \psi = f^{(k-1,n)}(\psi)$. Furthermore we have from the part above and by definition of $v^{(\cdot,n)}$ (see Remark 12) that for all $\psi \geq 1$

$$f^{(k,n)}(\psi) = v^{(k,n)}(\psi) \geq v^{(k-1,n)}(\psi) = f^{(k-1,n)}(\psi).$$

Combining these two inequalities shows that for all $x \geq \varphi^{(k,n)}$

$$\int_1^x \left(\beta_2 \xi^{-\beta_2} - \beta_1 \xi^{-\beta_1}\right) \frac{f^{(k,n)}(\xi)}{\xi}\,\mathrm{d}\xi > \int_1^x \left(\beta_2 \xi^{-\beta_2} - \beta_1 \xi^{-\beta_1}\right) \frac{f^{(k-1,n)}(\xi)}{\xi}\,\mathrm{d}\xi.$$

Now, recall the operator F_f defined in equation (23). With this notation and using the uniqueness of solutions we proved in the part above, we have that the unique root of $F_{f_{k-1}}$ determines $\varphi^{(k,n)}$ and the unique root of F_{f_k} determines $\varphi^{(k+1,n)}$. By construction we have $F_{f_{k-1}}(\varphi^{(k,n)}) = 0$ and the above inequality shows that for all $x \geq \varphi^{(k,n)}$ it follows $F_{f_k}(x) > F_{f_{k-1}}(x)$. Recalling that $F_{\cdot}(1) = 1$ we have as a consequence that the unique root of F_{f_k}, which equals $\varphi^{(k+1,n)}$, is strictly bigger than $\varphi^{(k,n)}$. ■

Justification for Algorithm 4. The function $v^{(k,n)}$ characterizes the value of the n-th order randomization of the optimal stopping problem at hand during the k-th exponential period. The expression given in Algorithm 4 is the function which is equal to $v^{(k,n)}$ over the time interval $((k-1)T/n, kT/n]$ rather than over the k-th exponential period. Lemma 19 below shows that there is pointwise convergence of $(v^{(k,n)}, b^{(k,n)})$ to (v, b) along an appropriate sequence of (k, n). ■

Lemma 19 *Let $k(n)$ be a sequence in n such that $k(n)T/n \to u$ as n tends to infinity. Then $(v^{(k(n),n)}(\psi), b^{(k(n),n)}(u))$ converges pointwise to $(v(\psi, u), b(u))$.*

Proof. We give the proof for the case $\alpha = 0$. The proof for $\alpha > 0$ follows with minor adjustments. Note in addition, for $\alpha > 0$ one may re-consider the proof of Proposition 5.1 of Kyprianou and Pistorius (2003) and note that with minor changes it also delivers the required convergence.

From the properties mentioned in Theorems 3 and 15 its easily deduced that $v^{(k(n),n)}(\psi) \to v(\psi, u)$ for all $\psi \geq 1$ implies $b^{(k(n),n)}(u) \to b(u)$, so we focus on proving the former convergence.

First a preliminary estimation. Note that we can write for all $n \geq 2$ and $\psi \geq 1$

$$
\begin{aligned}
v(\psi, n) - v(\psi, n-1) &\leq \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbf{E}_\psi \left(\Psi_{\tau \wedge n} - \Psi_{\tau \wedge (n-1)} \right) \\
&= \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbf{E}_\psi \left(\mathbf{E}_{\Psi_{n-1}} \left(\Psi_{\tau \wedge 1} \right) - \Psi_{n-1} \right) \\
&\leq \mathbf{E}_\psi \left(v(\Psi_{n-1}, 1) \right) \\
&\leq \mathbf{E}_\psi \left(\Psi_{n-1} \vee b(1) \right) \\
&\leq v(\psi, n-1) + b(1),
\end{aligned}
$$

where we used the strong Markov property and the properties of $v(.,1)$ and $b(.)$. It follows that for all $k, i \geq 1$ we can roughly estimate

$$
v(\psi, k+i) \leq 2^i (v(\psi, k) + b(1)). \tag{24}
$$

Now fix some $\psi \geq 1$ and $\epsilon > 0$. We use the notation from Remark 12, that is

$$
v^{(k(n),n)}(\psi) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathcal{E} \otimes \mathbf{E}_\psi \left(\Psi_{\tau \wedge \Theta_{k(n),n}} \right),
$$

and set $A_n := \{\Theta_{k(n),n} > u + \epsilon\}$ for every n. We can estimate

$$\begin{aligned}\left|v^{(k(n),n)}(\psi) - v(\psi,u)\right| \leq &\left|\sup_{\tau\in\mathcal{T}_{0,\infty}} \mathcal{E}\otimes\mathbf{E}_\psi\left(\mathbf{1}_{A_n^c}\Psi_{\tau\wedge\Theta_{k(n),n}}\right) - \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathbf{E}_\psi\left(\Psi_{\tau\wedge u}\right)\right| \\ &+ \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathcal{E}\otimes\mathbf{E}_\psi\left(\mathbf{1}_{A_n}\Psi_{\tau\wedge\Theta_{k(n),n}}\right)\end{aligned}$$

and the first term on the right is easily seen to converge to 0, for example by using dominated convergence after reformulating the expression under the measure $\mathbb{P}_1$ (see also equation (6)). With the notation

$$B_i^n := \left\{u + \epsilon + i - 1 < \Theta_{k(n),n} \leq u + \epsilon + i\right\}$$

for every n and $i \geq 1$, the second term allows the following estimate:

$$\begin{aligned}\sup_{\tau\in\mathcal{T}_{0,\infty}} \mathcal{E}\otimes\mathbf{E}_\psi\left(\mathbf{1}_{A_n}\Psi_{\tau\wedge\Theta_{k(n),n}}\right) &= \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathcal{E}\otimes\mathbf{E}_\psi\left(\sum_{i\geq 1}\mathbf{1}_{B_i^n}\Psi_{\tau\wedge\Theta_{k(n),n}}\right) \\ &\leq \sum_{i\geq 1}\sup_{\tau\in\mathcal{T}_{0,\infty}} \mathcal{E}\otimes\mathbf{E}_\psi\left(\mathbf{1}_{B_i^n}\Psi_{\tau\wedge(u+\epsilon+i)}\right) \\ &\leq \left(v(\psi,u+\epsilon) + b(1)\right)\cdot\sum_{i\geq 1}2^i\mathcal{E}(\mathbf{1}_{B_i^n}),\end{aligned}$$

where the last inequality uses (24). Using the known density of $\Theta_{k(n),n}$ it is a straightforward exercise to show that the last summation above tends to 0 as $n \to \infty$ and the proof is completed. ■

5 Numerical results and implementation

See http://ismi.math.uni-frankfurt.de/vanSchaik/Russian/webpublRussian.html.

6 Conclusion

In parallel with Peskir (2004b) this paper offers a characterization of the finite expiry Russian option as the unique solution to a free boundary problem. Further, using Carr's idea of 'Canadization' we deduce an algorithm to approximate the solution to this free boundary value problem. The algorithm captures numerically all the expected behaviour from the optimal stopping problem represented in (4) and (6). That is to say the free boundary respects the following logic.

- The greater the value of r or α the greater punishment the holder experiences for waiting causing the exercise threshold to move more dramatically to the origin.
- The larger the expiry date of the contract, the more the solution behaves like the perpetual case (for $\alpha > 0$) for which the optimal strategy is to exercise once the process Ψ crosses a fixed threshold.
- The larger the value of σ the more volatile the underlying Brownian motion is and hence it experiences 'larger' excursions. This allows for the holder to feel more free about waiting longer resulting in a larger exercise threshold.

Appendix

The constants in Algorithm 4 and Lemma 17 are given in a recursive way by the following systems of equations. Suppose that the functions $v^{(j,n)}$ and thresholds $\widetilde{\psi}_{(j,n)}$ are known for all $j = 0, ..., k-1$.

First we show how the $a(.,.,k)$ and $b(.,.,k)$ can be determined directly.

The constants $a(.,i,k)$ and $b(.,i,k)$ for $1 \leq i \leq k$ can be defined by a backwards recursion over i from their predecessors in the following way:

- if $i = k$ there are no $a(.,i,k)$ and $b(.,i,k)$ present;
- if $i = k-1$ then we have only one of each:

$$a(1,i,k) = \frac{-\lambda}{\frac{\sigma^2}{2}(2\beta_1 - 1) - r} c(1,k-1,k-1),$$
$$b(1,i,k) = \frac{-\lambda}{\frac{\sigma^2}{2}(2\beta_2 - 1) - r} c(2,k-1,k-1);$$

- if $1 \leq i \leq k-2$, then for every i we have $a(m,i,k)$ and $b(m,i,k)$ to be determined, for $1 \leq m \leq k-i$. This is again done by a backwards recursion, this time over m. So we start out with $m = k-i$:

$$a(m,i,k) = \frac{-\lambda}{m(\frac{\sigma^2}{2}(2\beta_1 - 1) - r)} a(m-1,i,k-1),$$
$$b(m,i,k) = \frac{-\lambda}{m(\frac{\sigma^2}{2}(2\beta_2 - 1) - r)} b(m-1,i,k-1),$$

  followed by, for $m = k-i-1, ..., 2$:

$$a(m,i,k) = \frac{\frac{-\lambda}{m} a(m-1,i,k-1) - \frac{\sigma^2(m+1)}{2} a(m+1,i,k)}{\frac{\sigma^2}{2}(2\beta_1 - 1) - r},$$
$$b(m,i,k) = \frac{\frac{-\lambda}{m} b(m-1,i,k-1) - \frac{\sigma^2(m+1)}{2} b(m+1,i,k)}{\frac{\sigma^2}{2}(2\beta_2 - 1) - r},$$

and we conclude by defining the first two:

$$a(1,i,k) = \frac{-\lambda c(1,i,k-1) - \sigma^2 a(2,i,k)}{\frac{\sigma^2}{2}(2\beta_1 - 1) - r},$$
$$b(1,i,k) = \frac{-\lambda c(2,i,k-1) - \sigma^2 b(2,i,k)}{\frac{\sigma^2}{2}(2\beta_2 - 1) - r}.$$

The terms $c(1,i,k)$ and $c(2,i,k)$ for $1 \leq i \leq k$ together with the threshold $\widetilde{\psi}_{(k,n)} > \widetilde{\psi}_{(k-1,n)}$ are determined by the following conditions given in Lemma 17

(i) $v^{(k,n)\prime}(1^+) = 0$,
(ii) $v^{(k,n)\prime}\left(\widetilde{\psi}^-_{(j,n)}\right) = v^{(k,n)\prime}\left(\widetilde{\psi}^+_{(j,n)}\right)$ and $v^{(k,n)}\left(\widetilde{\psi}^-_{(j,n)}\right) = v^{(k,n)}\left(\widetilde{\psi}^+_{(j,n)}\right)$ for all $1 \leq j \leq k-1$,
(iii) $v^{(k,n)}\left(\widetilde{\psi}^-_{(k,n)}\right) = \widetilde{\psi}_{(k,n)}$ and $v^{(k,n)\prime}\left(\widetilde{\psi}^-_{(k,n)}\right) = 1$.

Since $a(.,.,k)$, $b(.,.,k)$ and $\widetilde{\psi}_{(i,n)}$ for $1 \leq i \leq k-1$ are known at this time, we can define for $1 \leq i \leq k-1$

$$\begin{aligned} C_i := \widetilde{\psi}^{\beta_1}_{(i,n)} &\left(\sum_{m=1}^{k-i-1} a(m,i+1,k) \log(\widetilde{\psi}_{(i,n)})^m - \sum_{m=1}^{k-i} a(m,i,k) \log(\widetilde{\psi}_{(i,n)})^m \right) \\ &+ \widetilde{\psi}^{\beta_2}_{(i,n)} \left(\sum_{m=1}^{k-i-1} b(m,i+1,k) \log(\widetilde{\psi}_{(i,n)})^m - \sum_{m=1}^{k-i} b(m,i,k) \log(\widetilde{\psi}_{(i,n)})^m \right), \end{aligned}$$

$$\begin{aligned} D_i := \widetilde{\psi}^{\beta_1 - 1}_{(i,n)} &\left(\sum_{m=1}^{k-i-1} \beta_1 a(m,i+1,k) \log(\widetilde{\psi}_{(i,n)})^m + m a(m,i+1,k) \log(\widetilde{\psi}_{(i,n)})^{m-1} \right. \\ &\left. - \sum_{m=1}^{k-i} \beta_1 a(m,i,k) \log(\widetilde{\psi}_{(i,n)})^m + m a(m,i,k) \log(\widetilde{\psi}_{(i,n)})^{m-1} \right) \\ &+ \widetilde{\psi}^{\beta_2 - 1}_{(i,n)} \left(\sum_{m=1}^{k-i-1} \beta_2 b(m,i+1,k) \log(\widetilde{\psi}_{(i,n)})^m + m b(m,i+1,k) \log(\widetilde{\psi}_{(i,n)})^{m-1} \right. \\ &\left. - \sum_{m=1}^{k-i} \beta_2 b(m,i,k) \log(\widetilde{\psi}_{(i,n)})^m + m b(m,i,k) \log(\widetilde{\psi}_{(i,n)})^{m-1} \right) \end{aligned}$$

and

$$K := \frac{1}{\beta_2 - \beta_1} \sum_{m=1}^{k-1} \left[D_m \left(\beta_1 \widetilde{\psi}_{(m,n)}^{1-\beta_1} - \beta_2 \widetilde{\psi}_{(m,n)}^{1-\beta_2} \right) + \beta_1 \beta_2 C_m \left(\widetilde{\psi}_{(m,n)}^{\beta_2} - \widetilde{\psi}_{(m,n)}^{1-\beta_1} \right) \right].$$

Now $\widetilde{\psi}_{(k,n)}$ is defined as the unique solution bigger than $\widetilde{\psi}_{(k-1,n)}$ to the equation

$$\begin{aligned} &\beta_1 \beta_2 (1-\delta) \left(\widetilde{\psi}_{(k,n)}^{\beta_2+1} - \widetilde{\psi}_{(k,n)}^{\beta_1+1} \right) + \beta_2 K \left(\widetilde{\psi}_{(k,n)}^{2\beta_2} - \widetilde{\psi}_{(k,n)}^{\beta_1+\beta_2} \right) = \\ &\qquad \left((1-\delta) \widetilde{\psi}_{(k,n)} - K \widetilde{\psi}_{(k,n)}^{\beta_1} \right) \left(\beta_1 \widetilde{\psi}_{(k,n)}^{\beta_2} - \beta_2 \widetilde{\psi}_{(k,n)}^{\beta_1} \right), \end{aligned} \tag{25}$$

and $c(1,k,k)$ and $c(2,k,k)$ solve the linear system

$$\begin{aligned} \beta_1 c(1,k,k) + \beta_2 c(2,k,k) &= K, \\ \beta_1 \widetilde{\psi}_{(k,n)}^{\beta_1-1} c(1,k,k) + \beta_2 \widetilde{\psi}_{(k,n)}^{\beta_2-1} c(2,k,k) + \delta &= 1. \end{aligned} \tag{26}$$

Finally, the remaining $c(1,i,k)$ and $c(2,i,k)$ for $1 \leq i \leq k-1$ can be found by a backwards recursion over i, at every step using the pair

$$\begin{aligned} &\widetilde{\psi}_{(i,n)}^{\beta_1} c(1,i,k) + \widetilde{\psi}_{(i,n)}^{\beta_2} c(2,i,k) - \widetilde{\psi}_{(i,n)}^{\beta_1} c(1,i+1,k) - \widetilde{\psi}_{(i,n)}^{\beta_2} c(2,i,k) = C_i, \\ &\quad \beta_1 \widetilde{\psi}_{(i,n)}^{\beta_1-1} c(1,i,k) + \beta_2 \widetilde{\psi}_{(i,n)}^{\beta_2-1} c(2,i,k) - \beta_1 \widetilde{\psi}_{(i,n)}^{\beta_1-1} c(1,i+1,k) \\ &\qquad\qquad - \beta_2 \widetilde{\psi}_{(i,n)}^{\beta_2-1} c(2,i,k) = D_i. \end{aligned} \tag{27}$$

Acknowledgement

The research for this paper began in June 2002. We are grateful to Goran Peskir for his remarks and discussion concerning some aspects of this paper and his own recent work on this topic. We would also like to thank him for bringing to our attention at the later stages of this publication, the work of Ekström (2004) which provides a third and complementary analysis of the finite expiry Russian option focusing on the behaviour of the exercise boundary close to expiry.

References

[1] Avram, F., Asmussen, S. and Pistorius, M.R. (2004a) American and Russian options under exponential phase-type Lévy models. *Stoch. Proc. Appl.* **109**, 79-111.

[2] Avram, F., Kyprianou, A.E. and Pistorius, M.R. (2004b) Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options. *Ann. Appl. Probab.* **14**, 215-238.

[3] Carr, P. (1998) Randomization and the American Put. *Rev. Fin. Studies* **11**, 597–626.

[4] Eisenbaum, N. (2000) Integration with respect to local time. *Potential. Anal.* ***13**, 303–328.*

[5] Eisenbaum, N. (2004) Local time-space stochastics calculus for Lévy processes. *Preprint.*

[6] Ekström, E. (2004) Russian options with finite time horizon. *J. Appl. Probab.* **41** 313–326.

[7] Föllmer, H. Protter, P. and Shiryayev, A.N. (1995) Quadratic covariation and an extention of Itô's formula. *Bernoulli* **1** 149–169.

[8] Friedman, A. (1976) *Stochastic differential equations and applications. Vol I.* Academic Press.

[9] Graversen, S.E. and Peskir, G. (1998) On the Russian option: the expected waiting time. *Theory Probab. Appl.* **42**, 416–425.

[10] Karatzas, I. and Shreve, S. (1991) *Methods of Mathematical Finance.* Springer.

[11] Kyprianou, A.E. and Pistorius, M.R. (2003) Perpetual options through fluctuation theory. *Ann. Appl. Probab.* **13**, 1077-1098.

[12] Lamberton, D. (1998) American Options. In *Statistics of Finance* eds D. Hand and S. D. Jacka. Arnold.

[13] Myneni, R. (1992) The pricing of the American option. *Ann.Appl.Probab.* **2**, 1-23.

[14] Peskir, G. (2004a) On the American option problem. *To appear in Mathematical Finance.*

[15] Peskir, G. (2004b) The Russian Option: Finite Horizon. *To appear in Finance and Stochastics.*

[16] Peskir, G. (2004c) A change-of-variable formula with local time on curves. *To appear in J. Theor. Probab.*

[17] Peskir, G. (2004d) A change-of-variable formula with local time on surfaces. *Submitted.*

[18] van Schaik, K. (2003) The finite expiry Russian option, a characterization of its value and a method for approximating it. *Doctoranders thesis, Utrecht University.*

[19] Shiryaev, A.N. (1978) *Optimal stopping Rules.* Springer Verlag, Berlin

[20] Shiryaev, A.N. and Kabanov, Yu. and Kramkov, D.O. and Melnikov, A.V. (1994) Toward the theory of options of both European and American types.II. Continuous time. *Theory Probab. Appl.* **39**, 61–102.

[21] Shepp, L.A. and Shiryaev, A.N. (1993) The Russian option: reduced regret. *Ann.Appl.Probab.* **3**, 603–631.

[22] Shepp, L.A. and Shiryaev, A.N. (1995) A new look at pricing of the "Russian option". *Theory Probab. Appl.* **39**, 103–119.