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On the density of exponential functionals of Lévy processes

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In this paper, we study the existence of the density associated with the exponential functional of the Lévy process ξ,

$$I_{\mathbf{e}_q} := \int_0^{\mathbf{e}_q} \mathrm{e}^{\xi_s}\,\mathrm{d}s,$$

where $\mathbf{e}_q$ is an independent exponential r.v. with parameter $q \geq 0$. In the case where ξ is the negative of a subordinator, we prove that the density of $I_{\mathbf{e}_q}$, here denoted by k, satisfies an integral equation that generalizes that reported by Carmona *et al.* [7]. Finally, when $q = 0$, we describe explicitly the asymptotic behavior at 0 of the density k when ξ is the negative of a subordinator and at ∞ when ξ is a spectrally positive Lévy process that drifts to $+\infty$.

Keywords: exponential functional; Lévy processes; self-similar Markov processes; subordinators

1. Introduction

A real-valued Lévy process is a stochastic process issued from the origin with stationary and independent increments and almost-sure right-continuous paths with left limits. For background on Lévy processes see, e.g., [1] and [23]. We write $\xi = (\xi_t, t \geq 0)$ for its trajectory and $\mathbb{P}$ for its law. The law $\mathbb{P}$ of a Lévy process is characterized by its one-time transition probabilities. In particular, there always exists a triple (a, σ^2, Π), where $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and Π is a measure on $\mathbb{R} \setminus \{0\}$, satisfying the integrability condition $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$, such that for $t \geq 0$ and $z \in \mathbb{R}$,

$$\mathbb{E}\big[\mathrm{e}^{\mathrm{i}z\xi_t}\big] = \exp\big\{-\Psi(z)t\big\}, \tag{1.1}$$

where

$$\Psi(z) = \mathrm{i}az + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} \big(1 - \mathrm{e}^{\mathrm{i}zx} + \mathrm{i}zx\mathbf{1}_{\{|x|<1\}}\big)\Pi(\mathrm{d}x).$$

In the case when ξ is a subordinator, the Lévy measure Π has support on $[0, \infty)$ and fulfills the extra condition $\int_{(0,\infty)}(1 \wedge x)\Pi(\mathrm{d}x) < \infty$. Thus, the characteristic exponent Ψ can be expressed as

$$\Psi(z) = -\mathrm{i}cz + \int_{(0,\infty)} \big(1 - \mathrm{e}^{\mathrm{i}zx}\big)\Pi(\mathrm{d}x),$$

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where $c \geq 0$ and is known as the drift coefficient. It is well known that the function Ψ can be extended analytically on the complex upper half-plane, and so the Laplace exponent of ξ is given by

$$\phi(\lambda) := -\log \mathbb{E}\big[e^{-\lambda \xi_1}\big] = \Psi(i\lambda) = c\lambda + \int_{(0,\infty)} \big(1 - e^{-\lambda x}\big)\Pi(dx).$$

Similarly, in the case where ξ is a spectrally negative Lévy process (i.e., has no positive jumps), the Lévy measure Π has support on $(-\infty, 0)$, and the characteristic exponent Ψ can be written as

$$\Psi(z) = iaz + \frac{1}{2}\sigma^2 z^2 + \int_{(-\infty,0)} \big(1 - e^{izx} + izx\mathbf{1}_{\{x>-1\}}\big)\Pi(dx).$$

It is also well known that the function Ψ can be extended analytically on the complex lower half-plane, and so its Laplace exponent satisfies

$$\psi(\lambda) := \log \mathbb{E}\big[e^{\lambda \xi_1}\big] = -\Psi(-i\lambda) = a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} \big(e^{\lambda x} - 1 + -\lambda x\mathbf{1}_{\{x>-1\}}\big)\Pi(dx).$$

In this article, we examine the existence of the density associated with the exponential functional

$$I_{\mathbf{e}_q} := \int_0^{\mathbf{e}_q} e^{\xi_s}\, ds,$$

where $\mathbf{e}_q$ is an exponential random variable independent of the Lévy process ξ with parameter $q \geq 0$. If $q = 0$, then $\mathbf{e}_q$ is understood to be ∞. In this case, we assume that the process ξ drifts toward $-\infty$, because it is a necessary and sufficient condition for the almost-sure finiteness of $I := I_\infty$ (see, e.g., Theorem 1 of Bertoin and Yor [4]).

To the best of our knowledge, nothing is known about the existence of the density of $I_{\mathbf{e}_q}$ when $q > 0$. In the case where $q = 0$, the existence of the density of I has been proven by Carmona *et al.* [7] for Lévy processes with a jump structure of finite variation and recently by Bertoin *et al.* [2], Theorem 3.9, for any real-valued Lévy process. In particular, when ξ is the negative of a subordinator such that $\mathbb{E}[|\xi_1|] < \infty$, Carmona *et al.* [7], Proposition 2.1, proved that the random variable I has a density, k, that is the unique (up to a multiplicative constant) L^1-positive solution to the equation

$$(1 - cx)k(x) = \int_x^\infty \overline{\Pi}\big(\log(y/x)\big)k(y)\, dy, \qquad x \in (0, 1/c), \tag{1.2}$$

where $c \geq 0$ is the drift coefficient and $\overline{\Pi}(x) := \Pi(x, \infty)$. Here we generalize the foregoing equation. Indeed, we establish an integral equation for the density of $I_{\mathbf{e}_q}$, $q \geq 0$, when ξ is the negative of a subordinator. We note that when $q = 0$, the condition $\mathbb{E}[|\xi_1|] < \infty$ is not essential for the existence of its density and the validity of (1.2).

Another interesting problem is determining the behavior of the density of the exponential functional I at 0 and at ∞. This problem was recently studied by Kuznetzov [13] for Lévy processes with rational Laplace exponent (at 0 and at ∞), by Kuznetsov and Pardo [15] for hypergeometric Lévy processes (at 0 and at ∞), and by Patie [20] for spectrally negative Lévy processes (at ∞). In most applications, it is sufficient to have estimates of the tail behavior $\mathbb{P}(I \le t)$ when t goes to 0 and/or $\mathbb{P}(I \ge t)$ when t goes to ∞. The tail behavior $\mathbb{P}(I \le t)$ was studied by Pardo [19] in the case where the underlying Lévy process is spectrally positive and its Laplace exponent is regularly varying at infinity with index $\gamma \in (1,2)$, and by Caballero and Rivero [6] in the case when ξ is the negative of a subordinator whose Laplace exponent is regularly varying at 0. The tail behavior $\mathbb{P}(I \ge t)$ also has been studied in a general setting (see [8,18,21,22]). The second main result of this paper is related to this problem. Namely, we describe in detail the asymptotic behavior at 0 of the density of I when ξ is a subordinator, which in particular implies the behavior of $\mathbb{P}(I < t)$ near 0.

The paper is organized as follows. In Section 2 we state our main results. In particular, we study the density of $I_{\mathbf{e}_q}$ and the asymptotic behavior at 0 of the density of the exponential functional associated with the negative of a subordinator. In Section 3 we provide the proof of the main results, and in Section 4 we give some examples and some numerical results for the density of $I_{\mathbf{e}_q}$ when the driving process is the negative of a subordinator.

2. Main results

Our first main result states that $I_{\mathbf{e}_q}$ has a density for $q > 0$. Before we establish our first theorem, we introduce some notation and recall some facts about positive self-similar Markov processes (pssMp), which is our main tool in this first part.

Let $(\xi_t^\dagger, t \ge 0)$ be the process obtained by killing ξ at an independent exponential time of parameter $q > 0$, here denoted by $\mathbf{e}_q$. The law and the lifetime of $\xi^\dagger$ are denoted by $\mathbb{P}^\dagger$ and β, respectively.

We first note that

$$\left(I, \mathbb{P}^\dagger\right) = \left(\int_0^\beta \exp\left\{\xi_t^\dagger\right\} \mathrm{d}t, \mathbb{P}^\dagger\right) \stackrel{d}{=} \left(\int_0^{\mathbf{e}_q} \mathrm{e}^{\xi_t} \,\mathrm{d}t, \mathbb{P}\right).$$

For $x \ge 0$, let $\mathbb{Q}_x$ be the law of $X^{(x)}$, the positive self-similar Markov process with self-similarity index 1 issued from x associated with $\xi^\dagger$ via its Lamperti's representation (see [17] for more details on this representation), that is, for $x > 0$,

$$X_t^{(x)} = \begin{cases} x \exp\left\{\xi^\dagger_{\tau(t/x)}\right\}, & \text{if } \tau(t/x) < \infty, \\ 0, & \text{if } \tau(t/x) = \infty, \end{cases} \qquad t \ge 0,$$

where

$$\tau(s) = \inf\left\{r > 0\colon \int_0^r \mathrm{e}^{\xi_t^\dagger} \,\mathrm{d}t > s\right\}, \qquad \inf\{\varnothing\} = \infty$$

and 0 is a cemetery state. The process $X^{(x)}$ is a strong Markov process that fulfills the scaling property; that is, for $k > 0$,

$$\left(kX^{(x)}_{t/k}, t \ge 0\right) \stackrel{d}{=} \left(X_t^{(kx)}, t \ge 0\right).$$

We denote by $T_0^{(x)} := \inf\{t > 0\colon X_t^{(x)} = 0\}$, the first hitting time of $X^{(x)}$ at 0. Observe that for $s > 0$, we have the following equivalences:

$$\tau(s) < \infty \quad \text{iff} \quad \tau(s) \le \beta \quad \text{iff} \quad s \le \int_0^\beta \mathrm{e}^{\xi_t^\dagger}\,\mathrm{d}t.$$

Thus, from the construction of X, the following equality in law holds:

$$(T_0, \mathbb{Q}_1) \stackrel{d}{=} \left(\int_0^{\mathbf{e}_q} \mathrm{e}^{\xi_t}\,\mathrm{d}t, \mathbb{P}\right).$$

In what follows, we denote by $\mathbb{E}^{\mathbb{Q}_x}$ the expectation with respect to the probability measure $\mathbb{Q}_x$, $x \ge 0$.

We now have all of the elements necessary to establish our first main result. It concerns the existence of the density of $I_{\mathbf{e}_q}$.

Theorem 2.1. *Let $q > 0$. Then the function*

$$h(t) := q\mathbb{E}^{\mathbb{Q}_1}\left[\frac{1}{X_t}\mathbf{1}_{\{t<T_0\}}\right], \qquad t \ge 0,$$

is a density for the law of $I_{\mathbf{e}_q}$.

Corollary 2.2. *Assume that $q > 0$ and that ξ is a subordinator. Then the law of the random variable $I_{\mathbf{e}_q}$ is a mixture of exponentials; that is, its law has a density h on $(0, \infty)$ that is completely monotone. Furthermore, $\lim_{t\downarrow 0} h(t) = q$.*

In the sequel, we will assume that $\xi = -\zeta$, where ζ is a subordinator. We denote its drift by $c \ge 0$ and the renewal measure of the killed subordinator $(\zeta_t, t \le \mathbf{e}_q)$ by $U_q(\mathrm{d}x)$, that is,

$$\mathbb{E}\left[\int_0^{\mathbf{e}_q} f(\zeta_t)\,\mathrm{d}t\right] = \int_{[0,\infty)} f(x)U_q(\mathrm{d}x), \tag{2.1}$$

where f is a positive measurable function. If the renewal measure is absolutely continuous with respect to the Lebesgue measure, then the function $u_q(x) = U_q(\mathrm{d}x)/\mathrm{d}x$ is usually called the renewal density. If $q = 0$, then we denote U_0 and u_0 by U and u.

Our second main result generalizes the integral equation (1.2) of Carmona *et al.* for subordinators.

Theorem 2.3. *Let $q \ge 0$. The random variable $I_{\mathbf{e}_q}$ has a density that we denote by k, and it solves the equations*

$$\int_y^\infty k(x)\,\mathrm{d}x = \int_0^\infty k\big(y\mathrm{e}^x\big)U_q(\mathrm{d}x) \qquad \textit{almost everywhere}, \tag{2.2}$$

and

$$(1-cx)k(x)=\int_x^\infty \overline{\Pi}\big(\log(y/x)\big)k(y)\,\mathrm{d}y+q\int_x^\infty k(y)\,\mathrm{d}y,\qquad x\in(0,1/c). \tag{2.3}$$

Conversely, if a density on $(0,1/c)$ *satisfies any of the equations* (2.2) *or* (2.3), *then it is the density of* $I_{\mathbf{e}_q}$.

We illustrate the importance of the foregoing result in Theorem 2.5, where we study the asymptotic behavior at 0 of the density k, and in Section 4, where we provide some examples in which k can be computed explicitly. Further applications have been provided by Haas [11] and by Haas and Rivero [12], who used this equation to estimate the right tail behavior of the law of I and to study the maximum domain of attraction of I.

The following corollary is another important application of equation (2.3). In particular, it says that if we know the density of the exponential functional of the negative of a subordinator, say k, then for $\rho\geq 0$, $x^\rho k(x)$, adequately normalized is the density of the exponential functional associated to the negative of a new subordinator. The proof of this fact follows easily by multiplying in both sides of equation (2.3) by x^ρ. Such a result also has been given by Chazal *et al.* [9], but in terms of the distribution of $I_{\mathbf{e}_q}$, not in terms of its density.

Corollary 2.4. *Let* $q\geq 0$, $\rho>0$, *and* c_ρ *be the positive constants satisfying*

$$c_\rho=\int_{(0,\infty)} x^\rho k(x)\,\mathrm{d}x.$$

Then the function $h(x):=c_\rho^{-1}x^\rho k(x)$ *is the density of the exponential functional of the negative of a subordinator whose Laplace exponent is given by*

$$\phi_\rho(\lambda)=\frac{\lambda}{\lambda+\rho}\big(\phi(\lambda+\rho)+q\big). \tag{2.4}$$

Moreover, the density h *solves the equation*

$$(1-cx)h(x)=\int_x^\infty \overline{\Pi}_\rho(\log y/x)h(y)\,\mathrm{d}y,\qquad x\in(0,1/c), \tag{2.5}$$

where $\overline{\Pi}_\rho(z)=\overline{\Pi}(z)\mathrm{e}^{-\rho z}+q\mathrm{e}^{-\rho z}$.

We remark that the transformation studied by Chazal *et al.* [9] is more general than that presented in (2.4), and that they applied the transformation to Lévy processes with one-sided jumps. We also note that the subordinator with Laplace exponent given by ϕ_ρ has an infinite lifetime in any case.

Our next goal is to study the behavior of the density of $I_{\mathbf{e}_q}$ near 0. When $q=0$, we work with the following assumption:

(A) *The Lévy measure* Π *belongs to the class* $\mathcal{L}_\alpha$ *for some* $\alpha \geq 0$*; that is, the tail Lévy measure* $\overline{\Pi}$ *satisfies*

$$\lim_{x\to\infty} \frac{\overline{\Pi}(x+y)}{\overline{\Pi}(x)} = \mathrm{e}^{-\alpha y} \qquad \text{for all } y \in \mathbb{R}. \tag{2.6}$$

Observe that regularly varying and subexponential tail Lévy measures satisfy this assumption with $\alpha = 0$, and that convolution-equivalent Lévy measures are examples of Lévy measures satisfying (2.6) for some index $\alpha > 0$.

Theorem 2.5. *Let* $q \geq 0$ *and* $\xi = -\zeta$, *where* ζ *is a subordinator such that when* $q = 0$, *the Lévy measure* Π *satisfies assumption* (A). *The following asymptotic behavior holds for the density function* k *of the exponential functional* $I_{\mathbf{e}_q}$.

(i) *If* $q > 0$, *then*

$$k(x) \to q \qquad \textit{as } x \downarrow 0.$$

(ii) *If* $q = 0$, *then* $\mathbb{E}[I^{-\alpha}] < \infty$ *and*

$$k(x) \sim \mathbb{E}\big[I^{-\alpha}\big]\overline{\Pi}(\log 1/x) \qquad \textit{as } x \downarrow 0.$$

In the sequel, we will assume that $q = 0$. The foregoing result will help us describe the behavior at ∞ of the density of the exponential functional of a particular spectrally negative Lévy process associated with the subordinator ζ. To explain such relation, we need the following assumptions. Assume that U, the renewal measure of the subordinator ζ, is absolutely continuous with respect to the Lebesgue measure with density u, which is nonincreasing and convex. We also suppose that $\mathbb{E}[\zeta_1] < \infty$. According to Theorem 2 of Kyprianou and Rivero [16], there exists a spectrally negative Lévy process $Y = (Y_t, t \geq 0)$ that drifts to $+\infty$, with Laplace exponent described by

$$\psi(\lambda) = \lambda\phi^*(\lambda) = \frac{\lambda^2}{\phi(\lambda)} \qquad \text{for } \lambda \geq 0,$$

where ϕ^* is the Laplace exponent of another subordinator and satisfies

$$\phi^*(\lambda) := q^* + c^*\lambda + \int_{(0,\infty)} \big(1 - \mathrm{e}^{-\lambda x}\big)\Pi^*(\mathrm{d}x),$$

where

$$q^* = \left(c + \int_{(0,\infty)} x\Pi(\mathrm{d}x)\right)^{-1}, \qquad c^* = \begin{cases} 0, & c > 0 \text{ or } \Pi(0,\infty) = \infty, \\ 1/\Pi(0,\infty), & c = 0 \text{ and } \Pi(0,\infty) < \infty, \end{cases}$$

and the Lévy measure Π^* satisfies

$$U(\mathrm{d}x) = c^*\delta_0(\mathrm{d}x) + \big(q^* + \overline{\Pi}^*(x)\big)\mathrm{d}x \qquad \text{for } x \geq 0.$$

Let I_ψ be the exponential functional associated with $-Y$, that is,

$$I_\psi = \int_0^\infty \mathrm{e}^{-Y_s}\,\mathrm{d}s,$$

and denote its density by k_ψ. From the proof of Proposition 4 of Rivero [21], the density k_ψ satisfies

$$k_\psi(x) = q^* \frac{1}{x} k\left(\frac{1}{x}\right) \qquad \text{for } x > 0. \tag{2.7}$$

The following corollary explains the asymptotic behavior at ∞ of the density of the exponential functional of $-Y$.

Corollary 2.6. *Suppose that ζ is a subordinator satisfying assumption* (A) *such that its renewal measure has a density thst is nonincreasing and convex, and let Y be its associated spectrally negative Lévy process defined as above. Then the following asymptotic behavior holds for the density function k_ψ:*

$$k_\psi(x) \sim q^* \mathbb{E}\big[I^{-\alpha}\big] \frac{1}{x} \overline{\Pi}(\log x) \qquad \textit{as } x \to \infty.$$

3. Proofs

Proof of Theorem 2.1. We start the proof by showing that the function

$$h(t,x) := q\mathbb{E}^{\mathbb{Q}_x}\left[\frac{1}{X_t}\mathbf{1}_{\{t<T_0\}}\right], \qquad t \ge 0, x > 0,$$

is such that

$$\int_0^\infty h(t,x)\,\mathrm{d}t = 1 \qquad \text{for } x > 0. \tag{3.1}$$

Then the result follows from the identity (3.1) and the fact that

$$h(t+s) = \mathbb{E}^{\mathbb{Q}_1}\big[h(s, X_t)\mathbf{1}_{\{t<T_0\}}\big] \qquad \text{for } s, t \ge 0,$$

which is a straightforward consequence of the Markov property.

We now prove (3.1). From the definition of X and the change of variables $u = \tau(t/x)$, which implies that $\mathrm{d}u = x^{-1}\exp\{-\xi^\dagger_{\tau(t/x)}\}\,\mathrm{d}t$, we get

$$\begin{aligned}
&\int_0^\infty h(t,x)\,\mathrm{d}t \\
&\quad = q\int_0^\infty \mathrm{d}t\,\mathbb{E}\big[x^{-1}\exp\big\{-\xi^\dagger_{\tau(t/x)}\big\}\mathbf{1}_{\{\tau(t/x)<\infty\}}\big] \\
&\quad = q\mathbb{E}\left[\int_0^\infty x^{-1}\exp\big\{-\xi^\dagger_{\tau(t/x)}\big\}\mathbf{1}_{\{t \le x\int_0^\beta \mathrm{e}^{\xi^\dagger_s}\,\mathrm{d}s\}}\,\mathrm{d}t\right] = q\mathbb{E}\left[\int_0^\infty \mathbf{1}_{\{u\le\beta\}}\,\mathrm{d}u\right] = q\mathbb{E}(\beta) = 1.
\end{aligned}$$

We now prove that

$$\int_t^\infty h(s)\,\mathrm{d}s = \mathbb{P}(I_{\mathbf{e}_q} > t), \qquad t > 0.$$

Indeed, letting $t > 0$, making a change of variables, and using the semi-group property and Fubini's theorem, we have

$$\int_t^\infty h(s)\,\mathrm{d}s = \int_0^\infty h(s+t,1)\,\mathrm{d}s = \mathbb{E}^{\mathbb{Q}_1}\left[\left(\int_0^\infty h(s,X_t)\,\mathrm{d}s\right)\mathbf{1}_{\{t<T_0\}}\right] = \mathbb{Q}_1(t<T_0).$$

The result follows from the identity $\mathbb{Q}_1(t < T_0) = \mathbb{P}(I_{\mathbf{e}_q} > t)$. □

Proof of Corollary 2.2. Here we use the same notation as above and follow similar arguments as in the proofs of Lemma 5 and Proposition 1 of [3]. We first prove that for every $0 \le t < T_0$ and $p > 0$, the variable

$$X_t^p \int_t^{T_0} \frac{1}{X_s^{p+1}}\,\mathrm{d}s$$

is independent of $\sigma\{X_s, 0 \le s \le t\}$ and is distributed as

$$\int_0^{\mathbf{e}_q} \mathrm{e}^{-p\xi_s}\,\mathrm{d}s.$$

As a consequence of the Markov property at time t, we need only to show that under $\mathbb{Q}_x$, the variable

$$x^p \int_0^{T_0} \frac{1}{X_s^{p+1}}\,\mathrm{d}s$$

is distributed as $\int_0^{\mathbf{e}_q} \mathrm{e}^{-p\xi_s}\,\mathrm{d}s$. Then the change of variables $t = \tau(s/x)$, $s = x\int_0^t \mathrm{e}^{\xi_u^\dagger}\,\mathrm{d}u$ yields

$$\begin{aligned} x^p \int_0^{T_0} \frac{1}{X_s^{p+1}}\,\mathrm{d}s &= x^{-1}\int_0^{T_0} \mathrm{e}^{-(p+1)\xi^\dagger_{\tau(s/x)}}\,\mathrm{d}s \\ &= \int_0^\beta \mathrm{e}^{-(p+1)\xi_t^\dagger}\mathrm{e}^{\xi_t^\dagger}\,\mathrm{d}t \\ &= \int_0^\beta \mathrm{e}^{-p\xi_t^\dagger}\,\mathrm{d}t, \end{aligned}$$

which implies the desired identity in law, because $(\xi_t^\dagger, 0 \le t \le \beta)$ and $(\xi_t, 0 \le t \le \mathbf{e}_q)$ have the same law. Thus, we have

$$\mathbb{E}^{\mathbb{Q}_1}\left[\int_t^{T_0} \frac{1}{X_s^{p+1}}\,\mathrm{d}s\right] = \frac{\mathbb{E}^{\mathbb{Q}_1}[X_t^{-p}; t < T_0]}{\phi(p)+q},$$

which implies that

$$\frac{\partial \mathbb{E}^{\mathbb{Q}_1}[X_t^{-p}; t < T_0]}{\partial t} = -\big(\phi(p) + q\big)\mathbb{E}^{\mathbb{Q}_1}\big[X_t^{-(p+1)}; t < T_0\big].$$

By iteration, we have that the function $t \mapsto \mathbb{E}^{\mathbb{Q}_1}[X_t^{-p}; t < T_0]$ is completely monotone and takes value 1 for $t = 0$. Thus, taking $p = 1$, we deduce that $h(t)$ is completely monotone on $(0, \infty)$, and that $\lim_{t \downarrow 0} h(t) = q$. Finally from Theorem 51.6 and Proposition 51.8 of [23], we have that the law of $I_{\mathbf{e}_q}$ is a mixture of exponentials. $\square$

Proof of Theorem 2.3. By Theorem 2.1 (when $q > 0$) and Theorem 3.9 of [2] (when $q = 0$), we know that there exists a density of $I_{\mathbf{e}_q}$ for $q \geq 0$, which we denote by h. Moreover, [7] proved that the positive integer moments of $I_{\mathbf{e}_q}$ satisfy the following recursive equation:

$$\mathbb{E}\big[I_{\mathbf{e}_q}^n\big] = \frac{n}{\phi(n) + q}\mathbb{E}\big[I_{\mathbf{e}_q}^{n-1}\big], \qquad n > 0. \tag{3.2}$$

In particular, we have

$$\mathbb{E}\big[I_{\mathbf{e}_q}^n\big] = \frac{n!}{\prod_{i=1}^n (q + \phi(i))}, \qquad n \geq 0, \tag{3.3}$$

where the product is understood as 1 when $n = 0$.

The proof of (2.2) follows from the identity (3.2). Indeed, on the one hand, it is clear that

$$\mathbb{E}\big[I_{\mathbf{e}_q}^n\big] = \int_0^\infty x^n k(x)\,\mathrm{d}x = n\int_0^\infty \mathrm{d}y\, y^{n-1}\int_y^\infty k(x)\,\mathrm{d}x.$$

On the other hand, from the identity (2.1) with $f(x) = \mathrm{e}^{-nx}$ and a change of variables, we get

$$\begin{aligned}
\frac{n}{\phi(n) + q}\mathbb{E}\big[I_{\mathbf{e}_q}^{n-1}\big] &= n\int_0^\infty U_q(\mathrm{d}x)\mathrm{e}^{-nx}\int_0^\infty y^{n-1}k(y)\,\mathrm{d}y \\
&= n\int_0^\infty U_q(\mathrm{d}x)\int_0^\infty y^{n-1}\mathrm{e}^{-nx}k(y)\,\mathrm{d}y \\
&= n\int_0^\infty U_q(\mathrm{d}x)\int_0^\infty z^{n-1}k\big(z\mathrm{e}^x\big)\,\mathrm{d}z \\
&= n\int_0^\infty \mathrm{d}z\, z^{n-1}\int_0^\infty k\big(z\mathrm{e}^x\big)U_q(\mathrm{d}x).
\end{aligned}$$

Then, putting the pieces together, we have

$$\int_0^\infty \mathrm{d}y\, y^{n-1}\int_y^\infty k(x)\,\mathrm{d}x = \int_0^\infty \mathrm{d}y\, y^{n-1}\int_0^\infty k\big(y\mathrm{e}^{-x}\big)U_q(\mathrm{d}x) \qquad \text{for } n > 0,$$

which implies the desired result because the density

$$y \mapsto \frac{1}{\mathbb{E}(I_{\mathbf{e}_q})} \int_y^\infty k(x)\,\mathrm{d}x,$$

is determined by its positive integer moments, which readily follows from the fact that k is so.

Now, we verify the equation (2.3). We first prove that the function $\widetilde{h}$: $(0,\infty) \to (0,\infty)$, defined via

$$\widetilde{h}(x) = \begin{cases} cxh(x) + \int_x^\infty \overline{\Pi}\big(\log(y/x)\big)h(y)\,\mathrm{d}y + q\int_x^\infty h(y)\,\mathrm{d}y, & \text{if } x \in (0, 1/c), \\ 0, & \text{elsewhere}, \end{cases}$$

is a density for the law of $I_{\mathbf{e}_q}$ and thus that $h = \widetilde{h}$ a.e. Then we prove that the equality (2.3) holds. To do so, it is sufficient to verify that

$$\int_0^\infty x^n \widetilde{h}(x)\,\mathrm{d}x = \frac{n!}{\prod_{i=1}^n (q + \phi(i))}, \qquad n \in \mathbb{N},$$

given that the law of $I_{\mathbf{e}_q}$ is determined by its positive integer moments. Indeed, elementary computations, identity (2.2), and the fact that

$$\int_0^\infty \mathrm{e}^{-\theta y} U_q(\mathrm{d}y) = \frac{1}{\phi(\theta) + q}, \qquad \theta \geq 0,$$

give that for any integer $n \geq 0$,

$$\begin{aligned}
\int_0^\infty x^n \widetilde{h}(x)\,\mathrm{d}x &= c\int_0^\infty \mathrm{d}x\, x^{n+1} h(x) + \int_0^\infty \mathrm{d}x x^n \int_x^\infty \mathrm{d}y \overline{\Pi}\big(\log(y/x)\big)h(y) \\
&\quad + q\int_0^\infty \mathrm{d}x\, x^n \int_0^\infty h\big(x\mathrm{e}^y\big)U_q(\mathrm{d}y) \\
&= \frac{n!(n+1)c}{\prod_{i=1}^{n+1}(q+\phi(i))} + \int_0^\infty \mathrm{d}y\, h(y) \int_0^y \mathrm{d}x\, x^n \overline{\Pi}\big(\log(y/x)\big) \\
&\quad + q\int_0^\infty U_q(\mathrm{d}y) \int_0^\infty \mathrm{d}x\, x^n h\big(x\mathrm{e}^y\big) \\
&= \frac{n!(n+1)c}{\prod_{i=1}^{n+1}(q+\phi(i))} + \int_0^\infty \mathrm{d}y\, h(y) y^{n+1} \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-(n+1)z}\overline{\Pi}(z) \\
&\quad + q\int_0^\infty U_q(\mathrm{d}y)\mathrm{e}^{-(n+1)y} \int_0^\infty \mathrm{d}z\, z^n h(z) \\
&= \frac{n!(n+1)c}{\prod_{i=1}^{n+1}(q+\phi(i))} + \frac{(n+1)!}{\prod_{i=1}^{n+1}(q+\phi(i))} \frac{\int_0^\infty (1-\mathrm{e}^{-(n+1)z})\Pi(\mathrm{d}z)}{n+1}
\end{aligned}$$

$$
\begin{aligned}
&+q\frac{n!}{\prod_{i=1}^{n}(q+\phi(i))}\int_0^\infty U_q(\mathrm{d}y)\mathrm{e}^{-(n+1)y}\\
&=\frac{n!}{\prod_{i=1}^{n}(q+\phi(i))}\frac{(n+1)c+\int_0^\infty(1-\mathrm{e}^{-(n+1)z})\Pi(\mathrm{d}z)+q}{q+\phi(n+1)}\\
&=\frac{n!}{\prod_{i=1}^{n}(q+\phi(i))}.
\end{aligned}
$$

Now, let $\mathcal{N}=\{x\in\mathbb{R}\colon\ h(x)\neq\widetilde{h}(x)\}$. By the foregoing arguments, we know that the Lebesgue measure of $\mathcal{N}$ is 0. Let $k\colon\ (0,\infty)\to(0,\infty)$ be the function defined by

$$
k(x)=\begin{cases} h(x), & \text{if } x\in\mathcal{N}^c,\\ \dfrac{1}{1-cx}\left(\displaystyle\int_x^\infty\overline{\Pi}\bigl(\log(y/x)\bigr)h(y)\,\mathrm{d}y+q\int_x^\infty h(y)\,\mathrm{d}y\right), & \text{if } x\in\mathcal{N}.\end{cases}
$$

We now prove that $k(x)$ satisfies equation (2.3) everywhere. If $x\in\mathcal{N}^c$, then we have that $k(x)=h(x)=\widetilde{h}(x)$, and thus equation (2.3) is verified. Indeed, if $x\in\mathcal{N}$, then we have the following equalities:

$$
\begin{aligned}
&cxk(x)+\int_x^\infty\overline{\Pi}\bigl(\log(y/x)\bigr)k(y)\,\mathrm{d}y+q\int_x^\infty k(y)\,\mathrm{d}y\\
&\quad=cxk(x)+\int_x^\infty\overline{\Pi}\bigl(\log(y/x)\bigr)k(y)\mathbf{1}_{\{y\in\mathcal{N}^c\}}\,\mathrm{d}y+q\int_x^\infty k(y)\mathbf{1}_{\{y\in\mathcal{N}^c\}}\,\mathrm{d}y\\
&\quad=cxk(x)+\int_x^\infty\overline{\Pi}\bigl(\log(y/x)\bigr)h(y)\mathbf{1}_{\{y\in\mathcal{N}^c\}}\,\mathrm{d}y+q\int_x^\infty h(y)\mathbf{1}_{\{y\in\mathcal{N}^c\}}\,\mathrm{d}y\\
&\quad=\frac{cx}{1-cx}\left(\int_x^\infty\overline{\Pi}\bigl(\log(y/x)\bigr)h(y)\,\mathrm{d}y+q\int_x^\infty h(y)\,\mathrm{d}y\right)\\
&\qquad+\int_x^\infty\overline{\Pi}\bigl(\log(y/x)\bigr)h(y)\,\mathrm{d}y+q\int_x^\infty h(y)\,\mathrm{d}y\\
&\quad=k(x).
\end{aligned}
$$

Conversely, if k is a density on $(0,1/c)$ satisfying equation (2.2) or (2.3), then from the foregoing computations, it is clear that k and $I_{\mathbf{e}_q}$ have the same positive integer moments. This implies that k is a density of the exponential functional $I_{\mathbf{e}_q}$. □

Proof of Theorem 2.5. The proof consists of three steps. First, we show that when $q=0$, $\mathbb{E}[I^{-\alpha}]<\infty$. Then, for $q\geq 0$, we obtain a technical estimate on the maximal growth of $k(x)$ as $x\downarrow 0$. Finally, we obtain the statement of the theorem.

Step 1. We assume that $q=0$ and prove that $\mathbb{E}[I^{-\alpha}]<\infty$. The case where $\alpha=0$ is obvious. For $\alpha\in(0,1)$, we have from Theorem 2 of [4] that there exists a random variable R, independent of ξ, such that $IR\stackrel{d}{=}\mathbf{e}$, where $\mathbf{e}$ follows a unit mean exponential distribution. Because $\mathbb{E}[\mathbf{e}^{-\alpha}]<\infty$, the result follows.

Finally, let $\alpha \geq 1$. With (2.3) and some standard computations, we find that

$$\begin{aligned}\int_0^\infty x^{-\beta-1}k(x)\,\mathrm{d}x &= c\int_0^\infty \mathrm{d}x\, x^{-\beta}k(x) + \int_0^\infty \mathrm{d}x\, x^{-\beta-1}\int_x^\infty \mathrm{d}y\,\overline{\Pi}\big(\log(y/x)\big)k(y) \\ &= c\mathbb{E}\big[I^{-\beta}\big] + \int_0^\infty \mathrm{d}y\, k(y)\int_0^y \mathrm{d}x\, x^{-\beta-1}\overline{\Pi}\big(\log(y/x)\big) \\ &= c\mathbb{E}\big[I^{-\beta}\big] + \int_0^\infty \mathrm{d}y\, y^{-\beta}k(y)\int_0^\infty \mathrm{d}u\,\mathrm{e}^{\beta u}\overline{\Pi}(u) \\ &= -\frac{1}{\beta}\mathbb{E}\big[I^{-\beta}\big]\bigg(-c\beta + \int_0^\infty \big(1-\mathrm{e}^{\beta z}\big)\Pi(\mathrm{d}z)\bigg),\end{aligned}$$

that is,

$$\mathbb{E}\big[I^{-\beta-1}\big] = \mathbb{E}\big[I^{-\beta}\big]\frac{\phi(-\beta)}{-\beta}, \tag{3.4}$$

where ϕ is the Laplace exponent of ξ, which can be extended to $(-\alpha, \infty)$ because, for $\beta < \alpha$,

$$\int_0^\infty \big(\mathrm{e}^{\beta u}-1\big)\Pi(\mathrm{d}u) = \beta\int_1^\infty \overline{\Pi}\big(\log(z)\big)z^{\beta-1}\,\mathrm{d}z < \infty. \tag{3.5}$$

To see that (3.5) holds, note that $\overline{\Pi}(\log(z))$ is regularly varying with index $-\alpha$ by (2.6). Thus, $\overline{\Pi}(\log z) = z^{-\alpha}\ell(z)$ for a slowly varying function ℓ, and we can apply Proposition 1.5.10 of Bingham *et al.* [5].

Now, by iteratively using (3.4), we see that for $\mathbb{E}[I^{-\alpha}] < \infty$, it is sufficient to have $\mathbb{E}[I^{-\alpha'}] < \infty$ for some $\alpha' \in [0, 1)$. But this obviously holds if $\alpha' = 0$, whereas if $\alpha' \in (0, 1)$, it then holds by the same argument as used above for the case where $\alpha \in (0, 1)$.

Step 2. We assume that $q \geq 0$. For $q = 0$, let p be any function such that $p(0) = 0$ and $\min\{\theta - 1, 0\} < p(\theta) < \theta$, for all $\theta > 0$. When $q > 0$, the function p will be taken as 0, and thus the symbol $p(x)$ will be taken as 0. The goal of this step is to show that

$$\frac{k(x)}{x^{p(\alpha)}} \qquad \text{stays bounded as } x \downarrow 0, \tag{3.6}$$

where α is the parameter given in the assumption (A).

Observe that when $q > 0$, it follows from (2.3) that $\liminf_{x\to 0} k(x) \geq q$. Set $h(x) := k(x)/x^{p(\alpha)}$. We can write (2.3) as

$$1 - cx = x\int_1^\infty \overline{\Pi}\big(\log(z)\big)z^{p(\alpha)}\frac{h(xz)}{h(x)}\,\mathrm{d}z + \frac{qx^{p(\alpha)}\mathbb{P}(I_{\mathbf{e}_q} > x)}{h(x)}. \tag{3.7}$$

We argue by contradiction. Take some $\hat{x} \in (0, 1/c)$. If h were not bounded at $0+$, then $\mathbf{1}_{\{x\leq\hat{x}\}}h(x)$ would keep on attaining new maxima as $x \downarrow 0$. (Note that $\hat{x}$ is present just to ensure that this statement also holds if k is not bounded at $1/c-$.) In particular, this means that a sequence of points $(x_n)_{n\geq 0}$ exists with $x_n \downarrow 0$ as $n\to\infty$ and such that $h(x_n)\geq \sup_{x\in[x_n,\hat{x}]} h(x)$. We will show that this implies

$$x_n\int_1^\infty \overline{\Pi}\big(\log(z)\big)z^{p(\alpha)}\frac{h(x_n z)}{h(x_n)}\,\mathrm{d}z+\frac{q x_n^{p(\alpha)}\mathbb{P}(I_{\mathbf{e}_q}>x_n)}{h(x_n)}\to 0 \qquad \text{as } n\to\infty,$$

which indeed contradicts (3.7) because $1-cx_n\to 1$ as $n\to\infty$. Observe that if $q>0$ and h is not bounded at $0+$, then the second term in the latter equation tends to 0, because $p(\alpha)=0$ by construction. Thus, we just need to prove that the first term in the latter equation tends to 0. For this, we have

$$\begin{aligned} x_n\int_1^\infty \overline{\Pi}\big(\log(z)\big)z^{p(\alpha)}\frac{h(x_n z)}{h(x_n)}\,\mathrm{d}z &= x_n\int_1^{\hat{x}/x_n} \overline{\Pi}\big(\log(z)\big)z^{p(\alpha)}\frac{h(x_n z)}{h(x_n)}\,\mathrm{d}z \\ &\quad + x_n\int_{\hat{x}/x_n}^\infty \overline{\Pi}\big(\log(z)\big)z^{p(\alpha)}\frac{h(x_n z)}{h(x_n)}\,\mathrm{d}z. \end{aligned} \tag{3.8}$$

We first deal with the first integral on the right-hand side of (3.8). By construction of the sequence $(x_n)_{n\geq 0}$, we have $h(x_n z)\leq h(x_n)$ for any $z\in[1,\hat{x}/x_n]$; thus,

$$x_n\int_1^{\hat{x}/x_n} \overline{\Pi}\big(\log(z)\big)z^{p(\alpha)}\frac{h(x_n z)}{h(x_n)}\,\mathrm{d}z\leq x_n\int_1^{\hat{x}/x_n} \overline{\Pi}\big(\log(z)\big)z^{p(\alpha)}\,\mathrm{d}z. \tag{3.9}$$

If $q>0$ or $\alpha=0$ (recall $p(0)=0$), then we can take any $1<z_0$ and write

$$\begin{aligned} x_n\int_1^{\hat{x}/x_n} \overline{\Pi}\big(\log(z)\big)\,\mathrm{d}z &= x_n\int_1^{z_0} \overline{\Pi}\big(\log(z)\big)\,\mathrm{d}z + x_n\int_{z_0}^{\hat{x}/x_n} \overline{\Pi}\big(\log(z)\big)\,\mathrm{d}z \\ &\leq x_n\int_1^{z_0} \overline{\Pi}\big(\log(z)\big)\,\mathrm{d}z + x_n\left(\frac{\hat{x}}{x_n}-z_0\right)\overline{\Pi}\big(\log(z_0)\big), \end{aligned}$$

where the inequality uses that $\overline{\Pi}$ is decreasing. Letting $n\to\infty$, recalling that $x_n\downarrow 0$, we see that the first integral on the right-hand side vanishes, whereas the second term tends to $\hat{x}\overline{\Pi}(\log z_0)$. Because we can make this term arbitrarily small by choosing z_0 sufficiently large, because $\overline{\Pi}(\log z)\to 0$ as $z\to\infty$, it follows that (3.9) vanishes.

Next, consider the case where $\alpha>0$ and $q=0$. Because $\alpha-1<p(\alpha)<\alpha$, we can choose some $\beta\in(0,\alpha)$ such that $p(\alpha)-\beta+1\in(0,1)$. Using this, we find that

$$\begin{aligned} x_n\int_1^{\hat{x}/x_n} \overline{\Pi}\big(\log(z)\big)z^{p(\alpha)}\,\mathrm{d}z &= x_n\int_1^{\hat{x}/x_n} \overline{\Pi}\big(\log(z)\big)z^{\beta-1}z^{p(\alpha)-\beta+1}\,\mathrm{d}z \\ &\leq x_n\left(\frac{\hat{x}}{x_n}\right)^{p(\alpha)-\beta+1}\int_1^{\hat{x}/x_n} \overline{\Pi}\big(\log(z)\big)z^{\beta-1}\,\mathrm{d}z, \end{aligned}$$

and the right-hand side vanishes as $n\to\infty$, again because $x_n\downarrow 0$, and by (3.5).

It remains to show that the second integral on the right-hand side of (3.8) vanishes as $n \to \infty$. We have

$$x_n \int_{\hat{x}/x_n}^{\infty} \overline{\Pi}\big(\log(z)\big) z^{p(\alpha)} \frac{h(x_n z)}{h(x_n)} \,\mathrm{d}z \le x_n \overline{\Pi}\big(\log(\hat{x}/x_n)\big) \frac{1}{h(x_n)} \int_{\hat{x}/x_n}^{\infty} z^{p(\alpha)} h(x_n z) \,\mathrm{d}z$$
$$= \frac{\overline{\Pi}(\log(\hat{x}/x_n))}{x_n^{p(\alpha)}} \frac{1}{h(x_n)} \int_{\hat{x}}^{\infty} k(u) \,\mathrm{d}u,$$

where the inequality uses that $\overline{\Pi}$ is decreasing and to get the equality we apply the definition of h together with the substitution $u = x_n z$. Because k is a density and, by assumption, $h(x_n) \to \infty$ as n goes to ∞, for the right-hand side to vanish, it remains to show that $\overline{\Pi}(\log \hat{x}/x_n)/x_n^{p(\alpha)}$ stays bounded as n increases. When $q > 0$ or $\alpha = 0$ (recall that $p(0) = 0$), it is immediate, because $\overline{\Pi}$ is decreasing. When $\alpha > 0$ and $q = 0$, for any $1 < z_0 < z$, integration by parts yields

$$\overline{\Pi}\big(\log(z)\big) z^{p(\alpha)} = p(\alpha) \int_{z_0}^{z} \overline{\Pi}\big(\log(u)\big) u^{p(\alpha)-1} \,\mathrm{d}u + \int_{z_0}^{z} u^{p(\alpha)} \,\mathrm{d}\overline{\Pi}\big(\log(u)\big) + \overline{\Pi}\big(\log(z_0)\big) z_0^{p(\alpha)}.$$

Now, if we let z go to ∞, then, because $p(\alpha) < \alpha$, we see from (3.5) that the first integral on the right-hand side stays bounded, whereas the second integral is negative because $\overline{\Pi}$ is decreasing. Consequently, the left-hand side must stay bounded, and we are done.

Step 3, *case* $q = 0$. Denote $C_\alpha = \mathbb{E}[I^{-\alpha}]$, which is finite by step 1. From (2.3), we obtain, for all $x > 0$,

$$(1 - cx) \frac{k(x)}{\overline{\Pi}(\log(1/x))} = \int_x^{\infty} \frac{\overline{\Pi}(\log(y/x))}{\overline{\Pi}(\log(1/x))} k(y) \,\mathrm{d}y. \tag{3.10}$$

Because $\Pi(\log(1/x))$ is regularly varying (cf. (2.6)) with index $-\alpha$, we have that for any $\delta > 0$, there is D_δ such that $\Pi(\log(1/x)) \ge D_\delta x^{-\alpha-\delta}$ for x sufficiently small. Thus, the latter and property (3.6) imply that

$$\lim_{x \to 0} \frac{cxk(x)}{\overline{\Pi}(\log(1/x))} = 0. \tag{3.11}$$

Using equation (3.10) together with $k \ge 0$, Fatou's lemma, and identity (3.11) yields

$$\liminf_{x \downarrow 0} \frac{k(x)}{\overline{\Pi}(\log(1/x))} = \liminf_{x \downarrow 0} \frac{cxk(x)}{\overline{\Pi}(\log(1/x))} + \liminf_{x \downarrow 0} \int_x^{\infty} \frac{\overline{\Pi}(\log(y/x))}{\overline{\Pi}(\log(1/x))} k(y) \,\mathrm{d}y$$
$$\ge \int_0^{\infty} y^{-\alpha} k(y) \,\mathrm{d}y = C_\alpha.$$

In contrast, for any $\varepsilon > 0$, we have as $x \downarrow 0$,

$$\int_{\varepsilon}^{\infty} \frac{\overline{\Pi}(\log(y/x))}{\overline{\Pi}(\log(1/x))} k(y) \,\mathrm{d}y \to \int_{\varepsilon}^{\infty} y^{-\alpha} k(y) \,\mathrm{d}y \le C_\alpha.$$

If $\alpha > 0$, this follows from the fact that the convergence (2.6) is uniform over $y \in [\varepsilon, \infty)$ (see, e.g., Theorem 1.5.2 of [5]). If $\alpha = 0$ this uniformity holds only over intervals of the form $[\varepsilon, x_0]$, in which case we can write the left-hand side as the sum of integrals over $[\varepsilon, x_0]$ and $[x_0, \infty)$, the former in the limit again is bounded above by C_α, whereas for the latter, we can use that $\overline{\Pi}$ is decreasing to see

$$\int_{x_0}^{\infty} \frac{\overline{\Pi}(\log(y/x))}{\overline{\Pi}(\log(1/x))} k(y)\,\mathrm{d}y \leq \frac{\overline{\Pi}(\log(x_0/x))}{\overline{\Pi}(\log(1/x))} \int_{x_0}^{\infty} k(y)\,\mathrm{d}y,$$

then letting first $x \to \infty$, thereby using (2.6), and then $x_0 \to \infty$, it follows that this term vanishes.

So it remains to show that

$$\limsup_{x \downarrow 0} \int_x^{\varepsilon} \frac{\overline{\Pi}(\log(y/x))}{\overline{\Pi}(\log(1/x))} k(y)\,\mathrm{d}y \to 0 \qquad \text{as } \varepsilon \to 0.$$

For this, we get for ε small enough and $x < \varepsilon$,

$$\begin{aligned}
\frac{1}{\overline{\Pi}(\log(1/x))} \int_x^{\varepsilon} \overline{\Pi}\bigl(\log(y/x)\bigr) k(y)\,\mathrm{d}y &= \frac{x}{\overline{\Pi}(\log(1/x))} \int_1^{\varepsilon/x} \overline{\Pi}\bigl(\log(z)\bigr) k(xz)\,\mathrm{d}z \\
&\leq \frac{Cx}{\overline{\Pi}(\log(1/x))} \int_1^{\varepsilon/x} \overline{\Pi}\bigl(\log(z)\bigr) (xz)^{p(\alpha)}\,\mathrm{d}z \\
&= \frac{Cx^{1+p(\alpha)}}{\overline{\Pi}(\log(1/x))} \int_1^{\varepsilon/x} \overline{\Pi}\bigl(\log(z)\bigr) z^{p(\alpha)}\,\mathrm{d}z \\
&\sim \frac{C' x^{1+p(\alpha)}}{\overline{\Pi}(\log(1/x))} \left(\frac{\varepsilon}{x}\right)^{p(\alpha)+1} \overline{\Pi}\bigl(\log(\varepsilon/x)\bigr) \qquad \text{as } x \downarrow 0,
\end{aligned}$$

where C and C' are constants, the inequality holds by step 2 (cf. (3.6)), and the asymptotics follow from Karamata's theorem (see, e.g., Theorem 1.5.11 of [5]), which indeed applies here because $\overline{\Pi}(\log(z))$ is regularly varying with index $-\alpha$ (cf. (2.6)) and by construction (see step 2), $p(\alpha) \geq \alpha - 1$. Now, using (2.6), we see that ultimately, the right-hand side goes to $C' \varepsilon^{p(\alpha)+1-\alpha}$ as $x \downarrow 0$, and that this vanishes as $\varepsilon \to 0$ because, by construction, $p(\alpha) + 1 - \alpha > 0$ for all $\alpha \geq 0$.

Step 3, *case* $q > 0$. We will prove that

$$\int_x^{\infty} \overline{\Pi}\bigl(\log(y/x)\bigr) k(y)\,\mathrm{d}y \underset{x \to 0}{\to} 0.$$

By step 2, we can assume that k is bounded by $K \geq q$, in a neighborhood of $0+$. Letting $\delta > 1$ fixed, for x small enough, we have that

$$\begin{aligned}
\int_x^{x\delta} \overline{\Pi}\bigl(\log(y/x)\bigr) k(y)\,\mathrm{d}y &\leq K \int_x^{x\delta} \overline{\Pi}\bigl(\log(y/x)\bigr)\,\mathrm{d}y \\
&= K \int_0^{\log\delta} \overline{\Pi}(u) x \mathrm{e}^u\,\mathrm{d}u \leq K x \delta \int_0^{\log\delta} \overline{\Pi}(u)\,\mathrm{d}u \underset{x \to 0}{\to} 0.
\end{aligned}$$

In addition, we have that

$$\int_{x\delta}^{\infty} \overline{\Pi}\big(\log(y/x)\big)k(y)\,\mathrm{d}y \le \overline{\Pi}(\log\delta)\int_{x\delta}^{\infty} k(y)\,\mathrm{d}y \underset{x\to 0}{\to} \overline{\Pi}(\log\delta).$$

We conclude by making $\delta \to \infty$. Indeed, using equation (2.3) and the foregoing arguments, we conclude that

$$(1-cx)k(x) - q\mathbb{P}(I_{\mathbf{e}_q} > x) \underset{x\to 0}{\to} 0,$$

and the result follows. $\square$

4. Examples and some numerics

In this section, we illustrate Theorem 2.3, Corollary 2.4, and equation (2.7) with some examples, and provide some applications of Theorem 2.5.

Example 1. Let $q > 0$ and consider the case where the subordinator is just a linear drift with $c > 0$. By a simple Laplace inversion, we deduce $u_q(x) = c^{-1}\mathrm{e}^{-(q/c)x}$. Thus, from identities (2.3) and (2.2), we get

$$(1-cx)k(x) = \frac{q}{c}\int_{[0,\infty)} k\big(x\mathrm{e}^{y}\big)\mathrm{e}^{-(q/c)y}\,\mathrm{d}y, \qquad x \in (0, 1/c).$$

After straightforward computations, we deduce that the density of $I_{\mathbf{e}_q}$ is of the form

$$k(x) = q(1-cx)^{q/c-1}, \qquad x \in (0, 1/c).$$

It is important to note that we can get the density k by direct calculations, because

$$I_{\mathbf{e}_q} = \int_0^{\mathbf{e}_q} \mathrm{e}^{-ct}\,\mathrm{d}t = c^{-1}\big(1-\mathrm{e}^{-c\mathbf{e}_q}\big),$$

and $\mathbf{e}_q$ is exponentially distributed.

In what follows, we use the notation in Corollary 2.4 and in the discussion after Theorem 2.5. Let $\rho > 0$ and note that

$$\phi_\rho(\theta) = c\theta + q\frac{\theta}{\theta+\rho} \quad \text{and} \quad c_\rho = \frac{q}{c^{\rho+1}}\frac{\rho(\rho+1)\Gamma(q/c)}{\Gamma(\rho+q/c+1)}.$$

According to Corollary 2.4, the density of the exponential functional of the subordinator with Laplace exponent given by ϕ_ρ, satisfies

$$h(x) = c^{\rho+1}\frac{\Gamma(\rho+q/c+1)}{\Gamma(\rho+1)\Gamma(q/c)}x^{\rho}(1-cx)^{q/c-1} \qquad \text{for } x \in (0, 1/c),$$

in other words, the exponential functional has the same law as $c^{-1}B(\rho+1, q/c)$, where $B(\rho+1, q/c)$ is a beta random variable with parameters $(\rho+1, q/c)$.

We now consider the associated spectrally negative Levy process Y whose Laplace exponent is written as follows:

$$\psi(\lambda) = \frac{\lambda^2}{\phi_\rho(\lambda)} = \frac{\lambda(\lambda+\rho)}{c(\lambda+\rho)+q},$$

From (2.7), we deduce that the density of the exponential functional I_ψ associated to Y satisfies

$$k_\psi(x) = \frac{\rho c^{\rho+1}}{c\rho+q}\frac{\Gamma(\rho+q/c+1)}{\Gamma(\rho+1)\Gamma(q/c)}x^{-(\rho+q/c)}(x-c)^{q/c-1} \qquad \text{for } x > c.$$

Thus, I_ψ has the same law as $c(B(\rho, q/c))^{-1}$.

Example 2. Let $q = c = 0$, $\beta > 0$, and

$$\overline{\Pi}(z) = \frac{\beta}{\Gamma(a+1)}\mathrm{e}^{-((s-1)/a)z}\big(\mathrm{e}^{z/a}-1\big)^{a-1},$$

where $a \in (0, 1]$ and $s \geq a$. Thus, the Laplace exponent ϕ has the form

$$\phi(\theta) = \beta\frac{\theta\Gamma(a(\theta-1)+s)}{\Gamma(a\theta+s)}.$$

In this case, the equation (2.3) can be written as

$$\begin{aligned} k(x) &= \frac{\beta}{\Gamma(a+1)}\int_x^\infty (y/x)^{-(s-1)/a}\big((y/x)^{1/a}-1\big)^{a-1}k(y)\,\mathrm{d}y \\ &= \frac{\beta x}{\Gamma(a)}\int_0^\infty (z+1)^{a-s}z^{a-1}k\big(x(z+1)^a\big)\,\mathrm{d}z, \end{aligned}$$

where we are using the change of variable $z = (y/x)^{1/a} - 1$. After some computations, we deduce that

$$k(z) = \frac{\beta^{s/a}}{a\Gamma(s)}z^{(s-a)/a}\mathrm{e}^{-(\beta z)^{1/a}} \qquad \text{for } z \geq 0. \tag{4.1}$$

In other words, I has the same law as $\beta^{-1}\gamma_s^a$, where γ_s is a gamma random variable with parameter s.

If $a = 1$, then the process ξ is a compound Poisson process of parameter $\beta > 0$ with exponential jumps of mean $(s-1)^{-1} > 0$. From (4.1), it is clear that the law of its associated exponential functional has the same law as $\gamma_{(s,\beta)}$, a gamma random variable with parameters (s, β).

We now consider the associated spectrally negative Levy process Y with Laplace exponent satisfying

$$\psi(\lambda) = \frac{\lambda^2}{\phi(\lambda)} = \frac{\lambda\Gamma(a\lambda+s)}{\beta\Gamma(a(\lambda-1)+s)}.$$

The density of the exponential functional I_ψ associated with Y is given by

$$k_\psi(x) = \frac{\beta^{(s-a)/a}}{a\Gamma(s-a)} x^{-s/a} \mathrm{e}^{-(\beta/x)^{1/a}}, \qquad x > 0.$$

We remark that when $a = 1$, the process ξ is a Brownian motion with drift, and that the exponential functional I_ψ has the same law as $\gamma^{-1}_{(s-1,\beta)}$. This identity in law has been established by Dufresne [10].

Next, let $\rho > 0$ and note that

$$\phi_\rho(\theta) = \beta \frac{\theta\Gamma(a(\theta+\rho-1)+s)}{\Gamma(a(\theta+\rho)+s)} \quad \text{and} \quad c_\rho = \frac{\Gamma(a\rho+s)}{\beta^\rho\Gamma(s)}.$$

According to Corollary 2.4, the density of the exponential functional of the subordinator with Laplace exponent given by ϕ_ρ satisfies

$$h(x) = \frac{\beta^{(s+a\rho)/a}}{a\Gamma(a\rho+s)} x^{(a\rho+s-a)/a} \mathrm{e}^{-(\beta x)^{1/a}} \qquad \text{for } x > 0;$$

that is, it has the same law as $\beta^{-1}\gamma^a_{a\rho+s}$. In particular, the density of the exponential functional of its associated spectrally negative Levy process satisfies

$$k_\psi(x) = \frac{\beta^{(s+a\rho-a)/a}}{a\Gamma(a(\rho-1)+s)} x^{-(a\rho+s)/a} \mathrm{e}^{-(\beta/x)^{1/a}}, \qquad x > 0.$$

Example 3. Finally, let $a \in (0,1)$, $\beta \geq a$, $c = 0$, $q = \Gamma(\beta)/\Gamma(\beta-a)$,

$$\overline{\Pi}(z) = \frac{1}{\Gamma(1-a)} \int_z^\infty \frac{\mathrm{e}^{(1+a-\beta)x/a}}{(\mathrm{e}^{x/a}-1)^{1+a}} \,\mathrm{d}x \quad \text{and} \quad u_q(z) = \frac{1}{\Gamma(a+1)} \mathrm{e}^{-(\beta-1)z/a} \bigl(\mathrm{e}^{z/a}-1\bigr)^{a-1}.$$

The process ξ with such characteristics is a killed Lamperti stable subordinator with parameters $(1/\Gamma(1-a), 1+a-\beta, 1/a, a)$; see Section 3.2 in Kuznetsov *et al.* [14] for a proper definition. From Theorem 1.3, the density of $I_{\mathbf{e}_q}$ satisfies the equation

$$k(x) = \int_0^\infty \left(\frac{x\mathrm{e}^y}{\Gamma(1-a)} \int_z^\infty \frac{\mathrm{e}^{(1+a-\beta)x/a}}{(\mathrm{e}^{x/a}-1)^{1+a}} \,\mathrm{d}x + \frac{\Gamma(\beta)\mathrm{e}^{-(\beta-1)z/a}}{\Gamma(\beta-a)\Gamma(a+1)} \bigl(\mathrm{e}^{y/a}-1\bigr)^{a-1} \right) k\bigl(x\mathrm{e}^y\bigr) \,\mathrm{d}y.$$

Because the foregoing equation seems difficult to solve, we use the method of moments to determine the law of $I_{\mathbf{e}_q}$. We first note that

$$\mathbb{E}\bigl[I^n_{\mathbf{e}_q}\bigr] = \frac{n!\Gamma(\beta)}{\Gamma(an+\beta)},$$

and that in the case where $\beta = 1$, the exponential functional $I_{\mathbf{e}_q}$ has the same distribution as X_a^{-a}, where X_a is a α-stable positive random variable, that is,

$$\mathbb{E}\bigl[\mathrm{e}^{-\lambda X_a}\bigr] = \exp\bigl\{-\lambda^a\bigr\}, \qquad \lambda \geq 0;$$

see Section 3 in [3]. Recall that the negative moments of X_a are given by

$$\mathbb{E}\big[X_a^{-n}\big] = \frac{\Gamma(1+n/a)}{\Gamma(1+n)}, \qquad n \geq 0.$$

We now introduce $L_{(a,\beta)}$ and A, two independent random variables, whose laws are described as follows:

$$\mathbb{P}(L_{(a,\beta)} \in \mathrm{d}y) = \mathbb{E}\left[\frac{a\Gamma(\beta)}{\Gamma(\beta/a)X_a^{\beta}}; \frac{1}{X_a^a} \in \mathrm{d}y\right]$$

and

$$\mathbb{P}(A \in \mathrm{d}y) = (\beta/a - 1)(1-x)^{\beta/a-2}\mathbf{1}_{[0,1]}(x)\,\mathrm{d}x.$$

It is important to note from Example 1, that A has the same law as the exponential functional associated with the subordinator σ, which is defined as follows:

$$\sigma_t = t + \beta/a - 1, \qquad t \geq 0.$$

On the one hand, it is clear that

$$\mathbb{E}\big[L_{(a,\beta)}^n\big] = \frac{a\Gamma(\beta)}{\Gamma(\beta/a)}\mathbb{E}\big[X_a^{-(an+\beta)}\big] = \frac{\Gamma(\beta)}{\Gamma(\beta/a)}\frac{\Gamma(n+\beta/a)}{\Gamma(an+\beta)},$$

and on the other hand, we have

$$\mathbb{E}\big[A^n\big] = \frac{\Gamma(n+1)\Gamma(\beta/a)}{\Gamma(n+\beta/a)},$$

which implies that $I_{\mathbf{e}_q}$ has the same law as $L_{a,\beta}A$.

Finally, we numerically illustrate the density k and its asymptotic behavior at 0 for some particular subordinators ζ. First, we briefly discuss our method. Clearly, the equation (1.2) motivates the following straightforward discretization procedure. Approximate k by a step function $\tilde{k}$, that is,

$$\tilde{k}(x) = \sum_{i=0}^{N-1} \mathbf{1}_{\{x\in[x_i,x_{i+1})\}} y_i,$$

where $0 = x_0 < x_1 < \cdots < x_N = 1/c$ forms a grid on the x-axis. The heights y_i can then be found by iterating over $i = N-1, \ldots, 0$, thereby using (1.2) at each step, with $x = x_i$ and k replaced by $\tilde{k}$. Two remarks are pertinent here.

First, because (1.2) is linear in k, the condition that k is a density is required to uniquely determine the solution. This translates to the fact that the numerical procedure discussed above requires a starting point; that is, the value $y_{N-1} > 0$ should be known. (Of course, starting with $y_N = 0$ yields $\tilde{k} \equiv 0$.) We proceed by leaving y_{N-1} undetermined, running the iteration so that every y_i in fact becomes a linear function of y_{N-1}, and then finding y_{N-1} by requiring that $\tilde{k}$ integrates to 1.

Second, even though any choice of grid would work in principle, we found one particularly useful. Indeed, if we set $x_n = (1/c)\Delta^{N-n}$ for some Δ less than (but typically very close to) 1, then equation (2.3) yields the following relation:

$$\begin{aligned}(1 - cx_n)y_n &= \int_{x_n}^{\infty} \overline{\Pi}\bigl(\log(y/x_n)\bigr)\tilde{k}(y)\,\mathrm{d}y = x_n \int_1^{\infty} \overline{\Pi}\bigl(\log(z)\bigr)\tilde{k}(x_n z)\,\mathrm{d}z \\ &= x_n \sum_{i=n}^{N-1} y_i \int_1^{\infty} \overline{\Pi}\bigl(\log(z)\bigr)\mathbf{1}_{\{x_n z \in [x_i, x_{i+1})\}}\,\mathrm{d}z = x_n \sum_{i=n}^{N-1} y_i \int_{\Delta^{n-i}}^{\Delta^{n-i-1}} \overline{\Pi}\bigl(\log(z)\bigr)\,\mathrm{d}z.\end{aligned}$$

The approximation yielded by this setup is very efficient compared with, for example, the approximation using a standard equidistant grid, because in this case we need evaluate only N different integrals numerically[1].

We consider two examples in which the density k of I is explicitly known. The first example is taken from Example 2 with $a = 1$, $\beta = 2$, and $s = 3/2$. In this case, from (4.1), we have

$$k(x) = \frac{2^{5/2}}{\sqrt{\pi}} x^{1/2} \mathrm{e}^{-2x} \qquad \text{for } x > 0.$$

Figures 1–4 show plots of the density k, the difference $\tilde{k} - k$ (where $\tilde{k}$ is obtained by the foregoing method with $\Delta = 0.998$, yielding a grid of ~ 4500 points and a few minutes computation time on an average laptop), the ratio $k(x)/\overline{\Pi}(\log(1/x))$, and the ratio $\tilde{k}(x)/\overline{\Pi}(\log(1/x))$, respectively.

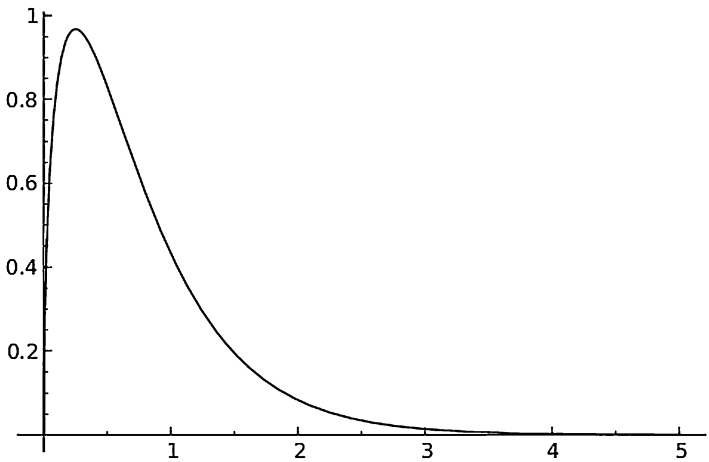

Figure 1. The density function k.

[1] All computations were done in the open source computer algebra system SAGE: www.sagemath.org

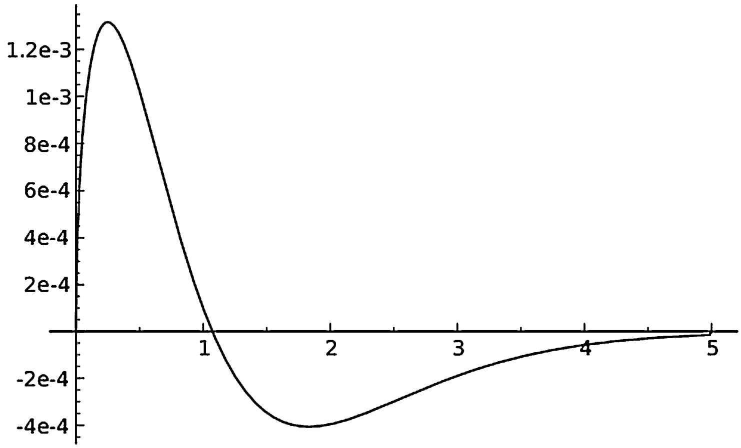

Figure 2. The difference $\tilde{k} - k$.

The second explicit example is also from Example 2 with $\beta = 1$ and $s = 1$ and $a = 1/2$. In this case, from (4.1), we have

$$k(x) = 2x\mathrm{e}^{-x^2} \qquad \text{for } x > 0.$$

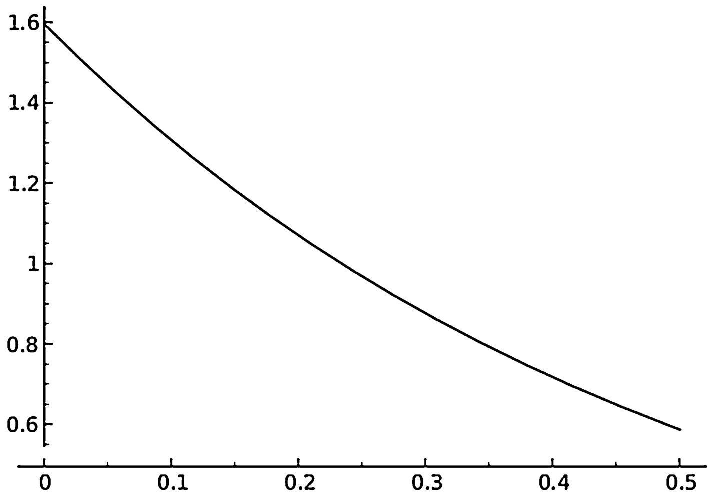

Figure 3. The ratio $k(x)/\overline{\Pi}(\log 1/x)$.

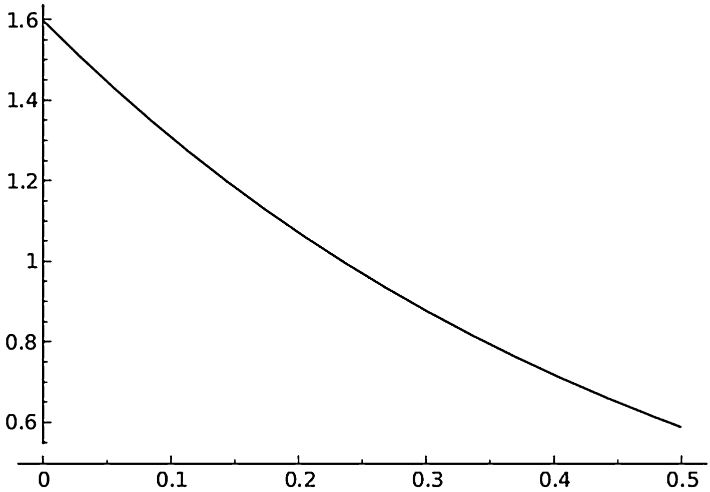

Figure 4. The ratio $\tilde{k}(x)/\overline{\Pi}(\log 1/x)$.

It is important to note that $\overline{\Pi}$ satisfies (A) with $\alpha = 1$. In this case, Figures 5–8 show plots plots of the density k, the difference $\tilde{k} - k$, the ratio $k(x)/\overline{\Pi}(\log(1/x))$, and the ratio $\tilde{k}(x)/\overline{\Pi}(\log(1/x))$, respectively.

We next examine two examples in which no formula for k is available. The first example is where ξ is a stable subordinator with drift, that is, $c = 1$ and $\Pi(\mathrm{d}x) = x^{-1-a}\,\mathrm{d}x$, where we take

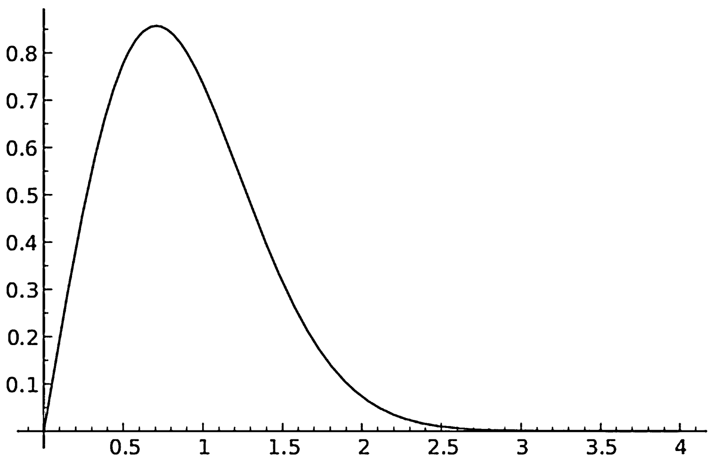

Figure 5. The density function k.

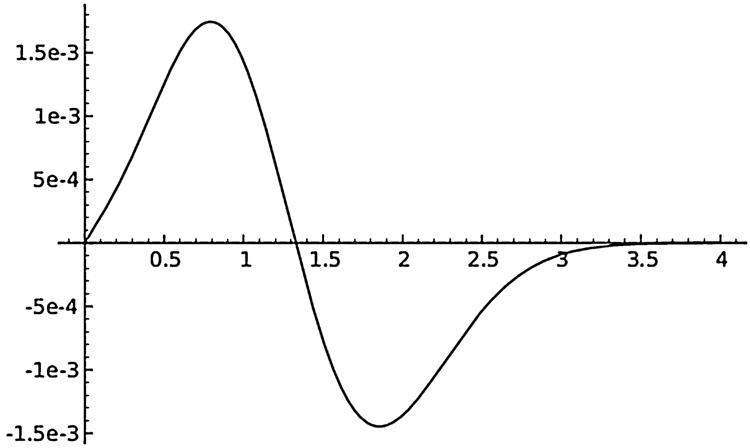

Figure 6. The difference $\tilde{k} - k$.

$a = 1/4$. Figures 9 and 10 show plots of $\tilde{k}$ and the ratio $\tilde{k}(x)/\overline{\Pi}(\log(1/x))$, respectively. Note that this is an example of a Lévy measure satisfying (2.6) with parameter 0.

The second example is a subordinator ξ with zero drift and Lévy measure of the form $\Pi(\mathrm{d}x) = x^{-1/4}\exp(-x^n)\,\mathrm{d}x$. Figure 11 shows $\tilde{k}$ for $n = 1$, $n = 2$, and $n = 3$. Figure 12 shows the ratio $\tilde{k}(x)/\overline{\Pi}(\log 1/x)$ for the case where $n = 1$, where (A) is satisfied with $\alpha = 1$.

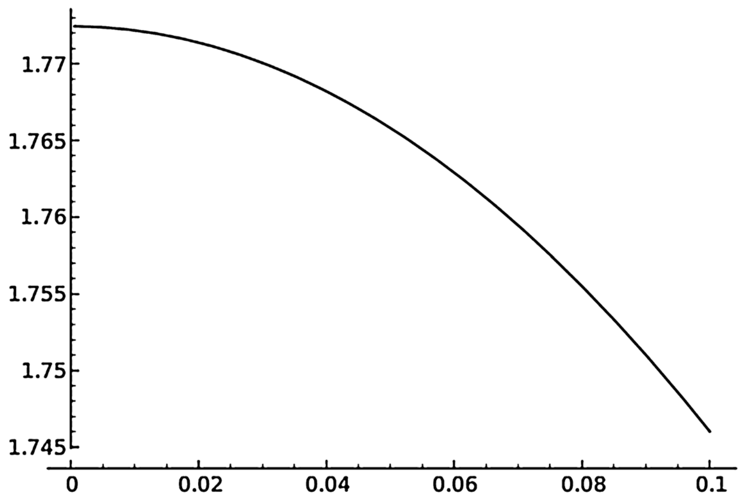

Figure 7. The ratio $k(x)/\overline{\Pi}(\log 1/x)$.

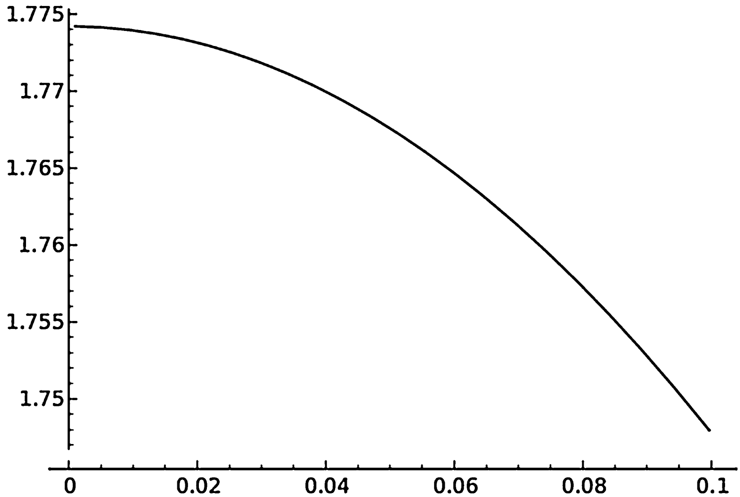

Figure 8. The ratio $\tilde{k}(x)/\overline{\Pi}(\log 1/x)$.

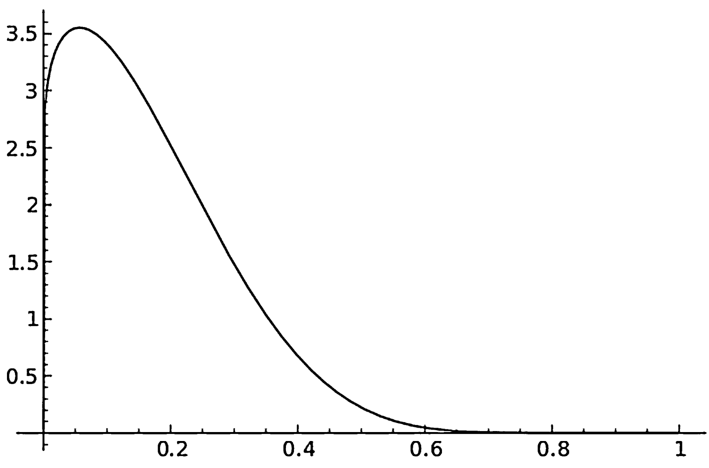

Figure 9. The density function $\tilde{k}$.

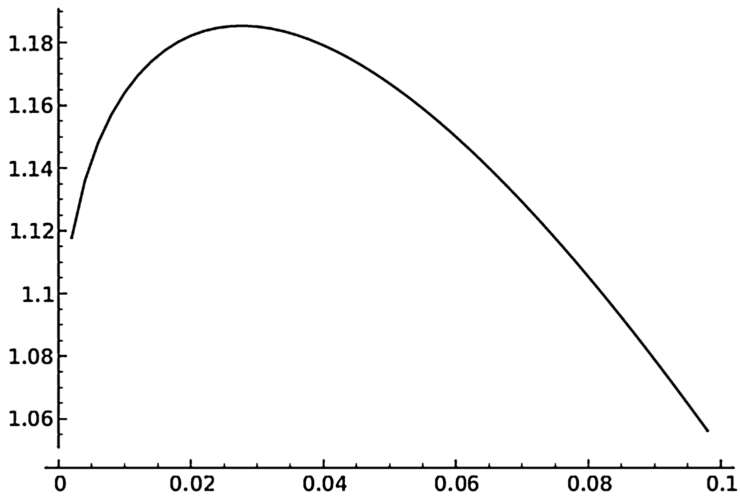

Figure 10. The ratio $\tilde{k}(x)/\overline{\Pi}(\log 1/x)$.

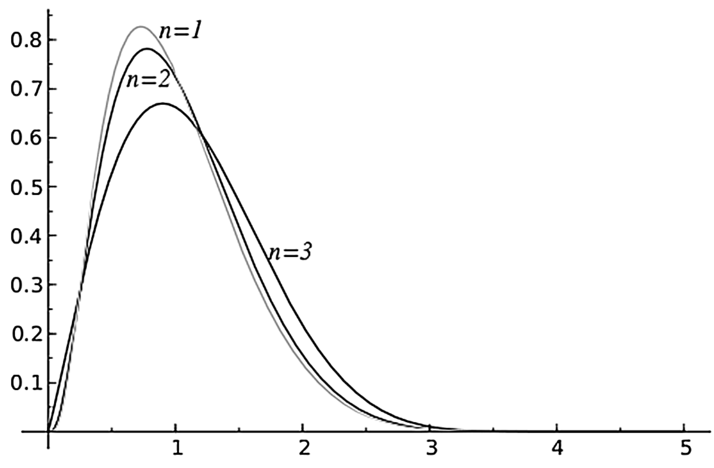

Figure 11. Density functions $\tilde{k}$.

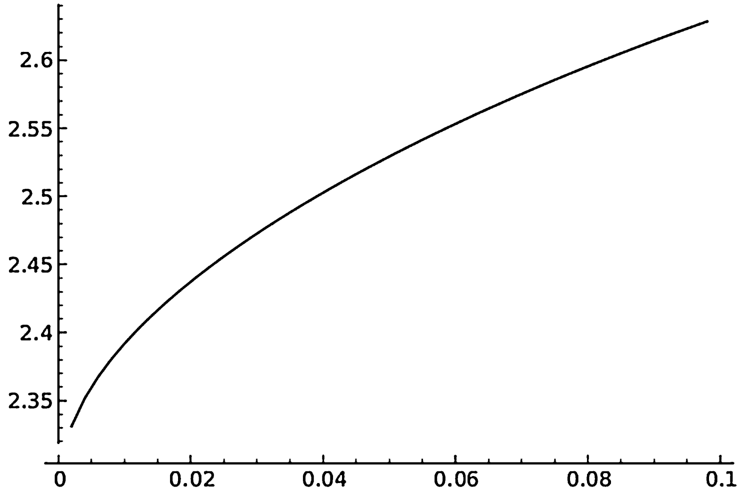

Figure 12. The ratio $\tilde{k}(x)/\overline{\Pi}(\log 1/x)$.

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